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Fundamental ideas of mathematics and the importance of affect-related aspects of learning

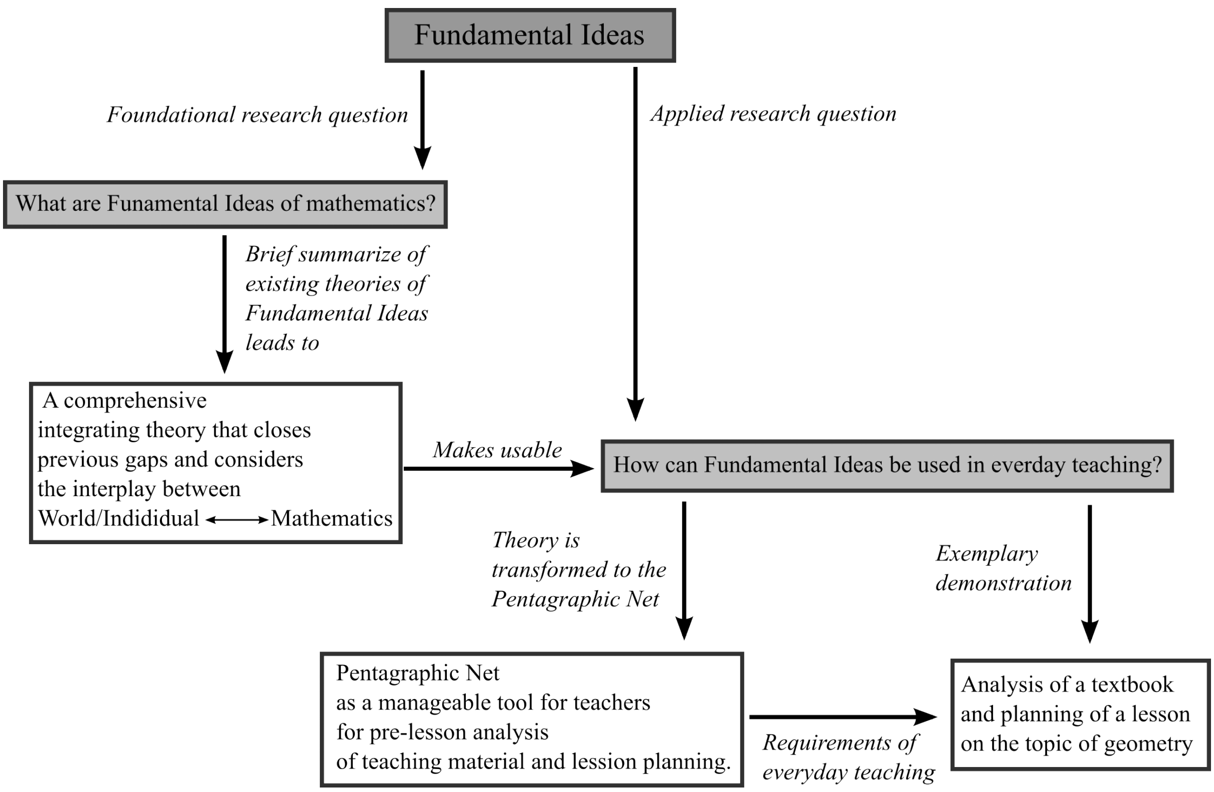
Marie-Christine von der Bank

Saarland University, Germany

Abstract: Since the development of the concept of Fundamental Ideas, they point out that learning mathematics in schools must involve more than merely presenting these ideas. In addition, we must enable our students to develop a positive attitude towards learning, including affect-related aspects that inspire a sense of excitement about discovery and self-confidence in one's abilities. In this paper, a theory of Fundamental Ideas is presented that combines and complements previous concepts. For implementing mathematical education in schools, this theory is reduced and solidified to its core. It can be used to analyse and create a learning setting focusing on affect-related aspects of learning and teaching mathematics. The usage is elucidated through an example, emphasising the planning of a lesson in the field of geometry.

Keywords: fundamental ideas, affect-related, emotion, belief, geometry

Correspondence: m.vonderbank@schule.saarland



1 Introduction

Joy, enthusiasm, curiosity, interest, intuition, creativity and persistence are essential to the pursuit of mathematics – they concern the essence of those engaged in mathematics (Hadamard, 1996).

In mathematics education, we should aim to transmit not only cognitive aspects (such as content knowledge and heuristics) to the learners but also affect-related aspects (such as scientific perspectives, attitudes, and dispositions). For engagement with mathematics, these are at least the seven aspects mentioned, which have been highlighted by great mathematicians, like Poincaré (1905), Hardy (1967), Hadamard (1996) or Wiles (2016), and surely each of us has experienced and felt them in our own engagement with mathematics. Unfortunately, affect-related aspects are often overlooked in lesson design at school. At least, they are seldom the starting point for planning a specific learning unit.

In order to address this, I propose two potential approaches. Firstly – and this is the main body of this paper – aligning instructional strategies with the Fundamental Ideas of mathematics, as originally stated by Bruner (1960). Fundamental Ideas can be defined as central aspects of mathematics and mathematical practices which recur across diverse subdomains, and therefore, provide a cohesive framework for interconnecting various content domains of mathematics in school. Although Bruner originally considered affect-related aspects in the exploration of Fundamental Ideas, these aspects are scarcely addressed in later research works. So as a first step, I will briefly outline a concept that refers to Fundamental Ideas as mathematical content, activities and also as specific attitudes. The aim of this paper then is to present and use the framework of the *Pentagraphic Net* (Figure 3), as a theoretical starting point for a discussion about how teachers, in different ways can evaluate teaching material or teaching settings while focusing on Fundamental Ideas. The Pentagraphic Net itself is a didactical tool that combines, concretise, and reduces the complexity of various concepts of Fundamental Ideas. As it also highlights affect-related aspects it can help to emphasise them from the planning of a lesson onwards.

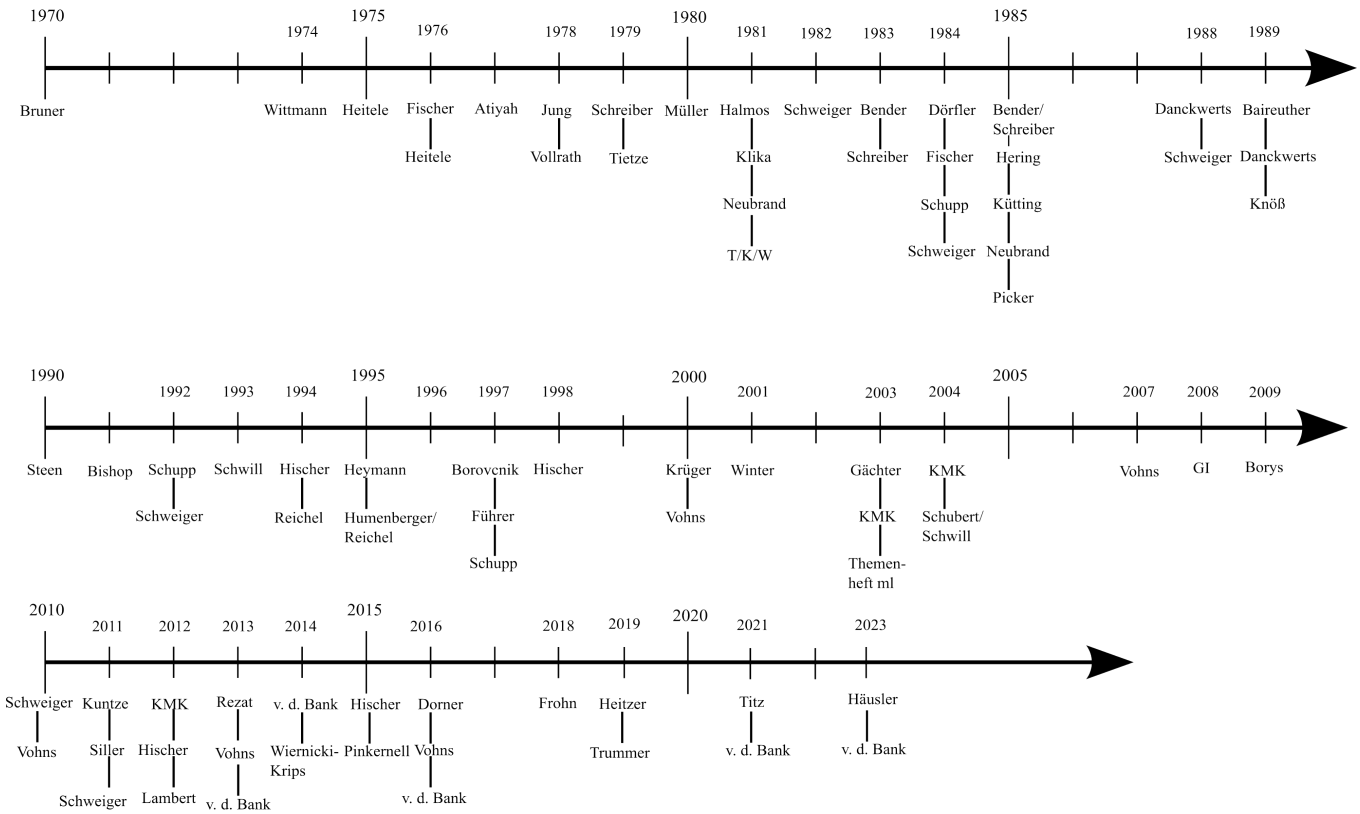
With the second approach, which will be just an outlook in this paper, the teachers themselves now come into focus. Their cognitive and affect-related experiences of engaging in mathematics, their attitudes, perspectives, beliefs about learning processes, and ultimately their commitment to their learners play a central role in instruction and also transfer to the learners (Frenzel & Götz, 2018). So, in order to develop teaching units that enable students to have a positive emotional experience with mathematics, the attitudes of teachers must also be taken into account.

2 Theory of Fundamental Ideas

When Bruner first developed the concept of Fundamental Ideas, he pointed out “that the curriculum should no longer consist solely of disconnected factual knowledge but should emphasise the core of the educational process, which is non-specific transfer” (Bruner, 1960, p. 18). This means that, at first, a general idea is learned, not a specific skill. This general idea then is used to recognise later problems as special cases of the concept. Bruner calls “general” ideas “fundamental” as “they lie at the heart of all science and mathematics” and have a “wide as well as powerful applicability” (Bruner, 1960, p. 19). Bruner also mentions that Fundamental Ideas involve affect-related aspects, like a specific “attitude towards learning and inquiry, towards guessing and hunches, towards the possibility of solving problems on one’s own” (Bruner, 1960, p. 20). With these aspects, he aims to cultivate “a sense of excitement about discovery [...] and a resulting sense of self confidence in one’s abilities” (Bruner, 1960, p. 20).

From a research point of view, this vague definition immediately raises the question: *What are the Fundamental Ideas of mathematics?* Bruner himself does not provide specific examples of Fundamental Ideas but aims to address this question collaboratively with experts in the respective fields and educators from practical teaching experience. In the past 60 years, numerous scholars from the fields of mathematics and didactics have responded to this call. Figure 1 provides an overview with a focus on the really strong research tradition in Germany-speaking countries. But certainly, the concept of Fundamental Ideas has played a role in other countries too. Examples are found in Halmos (1981), Steen (1990) and Bishop (1991), as well as in the “ABCMath”-project funded by the EU focused on designing teaching based on so-called “Big Ideas” (Kuntze et al., 2011).

Figure 1. Research works arranged on a timeline.

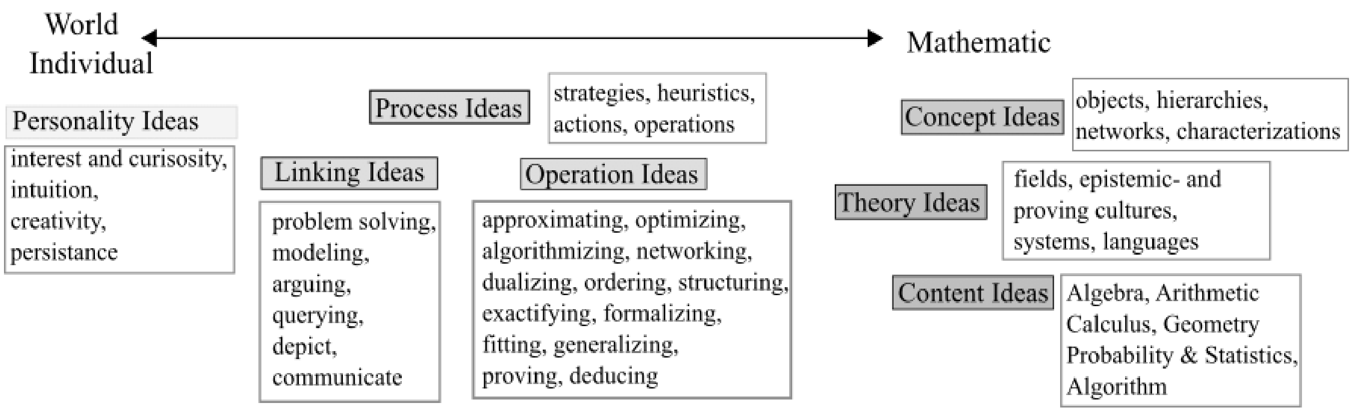


Consequently, there is now a substantial number of formal definitions and catalogues of Fundamental Ideas, which differ significantly from each other. An overview is given in (von der Bank, 2016), (von der Bank, 2023) and (Vohns, 2016). The range of ideas mentioned spans from, e.g., geometric series, quotient structure and eigenvalues (Halmos, 1981) as mathematical *content* to *activities* like locating, designing, playing and explaining (Bishop, 1991). Schweiger (2010) sees this diversity as an opportunity, particularly for instruction: "Finding an individual catalogue that can be revised repeatedly could be a rewarding part of the didactic reflection accompanying instruction" (Schweiger, 2010, p. 1).

Following this suggestion, I will now state my own understanding of Fundamental Ideas and then outline my catalogue of Fundamental Ideas. Due to limited space in this paper, I can only provide a brief summary. For a detailed rationale, please refer to (von der Bank, 2016). However, the objective in my design was to align with the general conceptualization (Wedman, 2020) and, therefore, to open up the formal and prototypical understanding of existing concepts, thus closing gaps that have existed so far. Due to its broad scope, my catalogue provides teachers with room for individual appropriation and setting of priorities in the classroom. My concept of Fundamental Ideas also explicitly takes into account attitudes that are typical for those engaged with mathematics. Therefore, I define *Fundamental Ideas* as central aspects of mathematics and mathematical activity, such as *content, actions,* and *attitudes*. It is their interplay that constitutes the essence of mathematics.

Similar to Bruner's original definition, mine is intentionally kept open to allow room for individual appropriation. This expanded conceptual understanding influences my catalogue of ideas (Figure 2), which integrates existing ones from research tradition and structures them more strongly into categories, namely *Concept Ideas, Theory Ideas* and *Content Ideas* which refer to the character of mathematic as a product. The process side of mathematics is described with the categories of *Process Ideas, Linking Ideas* and *Operation Ideas*. The catalogue explicitly considers (positive) attitudes towards inquiry through the category of *Personality Ideas*, thus covering the broad interplay between the World/Individual and Mathematics.

Figure 2. Fundamental Ideas in the interplay of World/Individual and Mathematics.



In this interplay, Concept Ideas, Content Ideas and Theory Ideas can be located on the side of mathematics. Concept Ideas and Content Ideas encompass mathematical content from various fields and ideas that are significant for mathematical concepts, such as the objects defining these concepts and their orders forming networks. These two categories of ideas thus serve a direct description and organization of mathematics. Fields and cultures of knowledge and justification, summarised in the Theory Ideas, however, do not primarily describe the content of mathematics but its framework. They entail historical changes in science and educational policy influences on mathematics. For example, it is subject to a constantly changing scientific zeitgeist to aggregate specific content into fields and an educational zeitgeist to select from the fields of mathematics those for mathematics instruction. Also, the demanded level of rigor in concept definitions or proofs can change. Therefore, the Theory Ideas are situated on a meta-level that influences both Concept Ideas and Content Ideas.

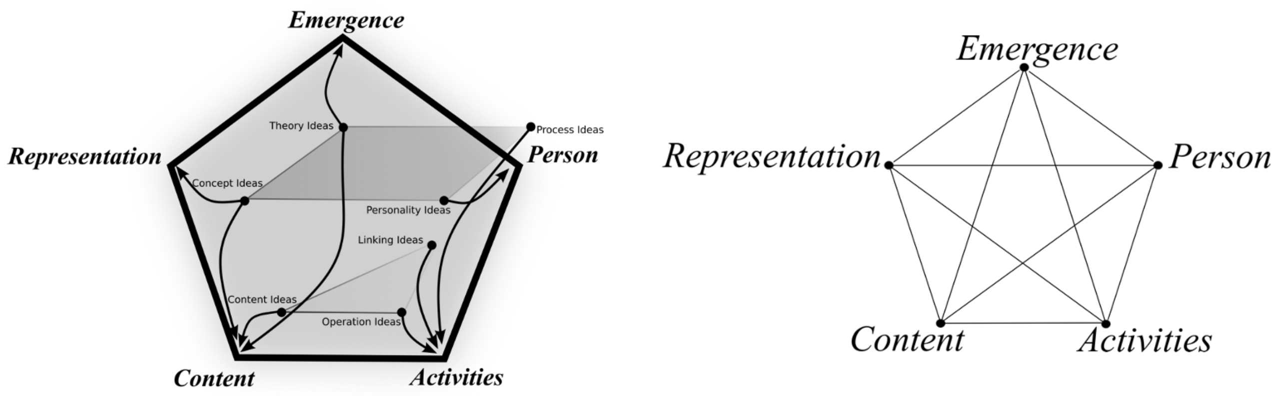
The direct interplay of the two sides is described in the above model through Process Ideas, Linking Ideas and Operation Ideas. The more abstract Process Ideas signify that mathematicians, when engaging with mathematics, develop typical approaches that can be categorised into strategies, heuristics, actions, and operations (as reversibly conceivable actions). Linking Ideas and Operation Ideas encompass the concrete interaction of people with mathematics. Linking Ideas tend to originate more from mathematics, in the sense that a person solves a specific mathematical problem and models, communicates, questions, etc. Operation Ideas are more reversed in direction. Here, the person acts on mathematics by executing operations with and on mathematical content (e.g., optimizing the value of a function).

With the category of Personality Ideas, the individual as a researching mathematician comes into focus. These ideas reflect personal attitudes that play a central role in engaging with mathematics and drive the research process forward. Without interest and curiosity, one would not even turn to a mathematical problem in the first place. In the process of finding solutions, intuition and creativity are just as important as domain-specific knowledge. Finally, persistence is required to solve mathematical problems. This is e.g., how Wiles (2016) characterised what is important for dealing with mathematics and what enabled him to prove Fermat's last theorem.

3 Fundamental Ideas in mathematics education

This theory is too complex to be used directly in mathematics instruction. It requires a reduction to become a manageable tool for teachers. Here the question of the essential aspects and functions of Fundamental Ideas in instruction becomes guiding. In line with the research tradition of Fundamental Ideas, these ideas serve the reasoned selection of learning material and the integration of relevant aspects of mathematics in teaching. Like Bruner stated Fundamental Ideas should facilitate multiple connections between aspects relevant to teaching. For mathematics instruction these relevant aspects can be visualised as nodes of this *Pentagraphic Net* in Figure 3 (right side). The nodes of the Pentagraphic Net are derived from the categories presented above, where the Fundamental Ideas mentioned there are organised, specified, and summarised. This process is indicated by arrows in Figure 3 (left side).

Figure 3. Reduced theory of Fundamental Ideas to the Pentagraphic Net.



While the process of transitioning from the categories of ideas to the nodes of the Pentagraphic Net is complex, I will not go into detail. But I would like to convey an idea of the procedure through some illustrative assignments.

From the concretisations of Process Ideas and the summary of rather application-oriented Linking Ideas, and rather intramathematical Operation Ideas, the node '*Activities*' is formed. The node '*Content*' encompasses the Content Ideas (as concretisations of mathematical fields), as well as the content aspects of the Concept Ideas, such as objects and their characterisation. Proofs also belong to this node as part of the Theory Ideas.

The use of different representations of objects and concepts, as well as important aspects of concept formation such as the distinction between visualisation and representation, and cognitive preferences and levels of representation, are derived from the Content and Concept Ideas, and they are summarised in the node '*Representation*'.

To convey an appropriate understanding of mathematics as a process and a product, suitable problems and solutions should also address their historical emergence. The influences of Theory Ideas on Concept Ideas and Content Ideas, as described above, are therefore implemented in the node '*Emergence*' for instruction. This is an important node of the Pentagraphic Net but it will not be the focus of its following demonstration in this paper.

Last, the node '*Person*' encompasses the ideas related to personality.

Even the edges of the Pentagraphic Net are meaningful. They visually represent relevant connections between two nodes.

4 Using the Pentagraphic Net in everyday teaching

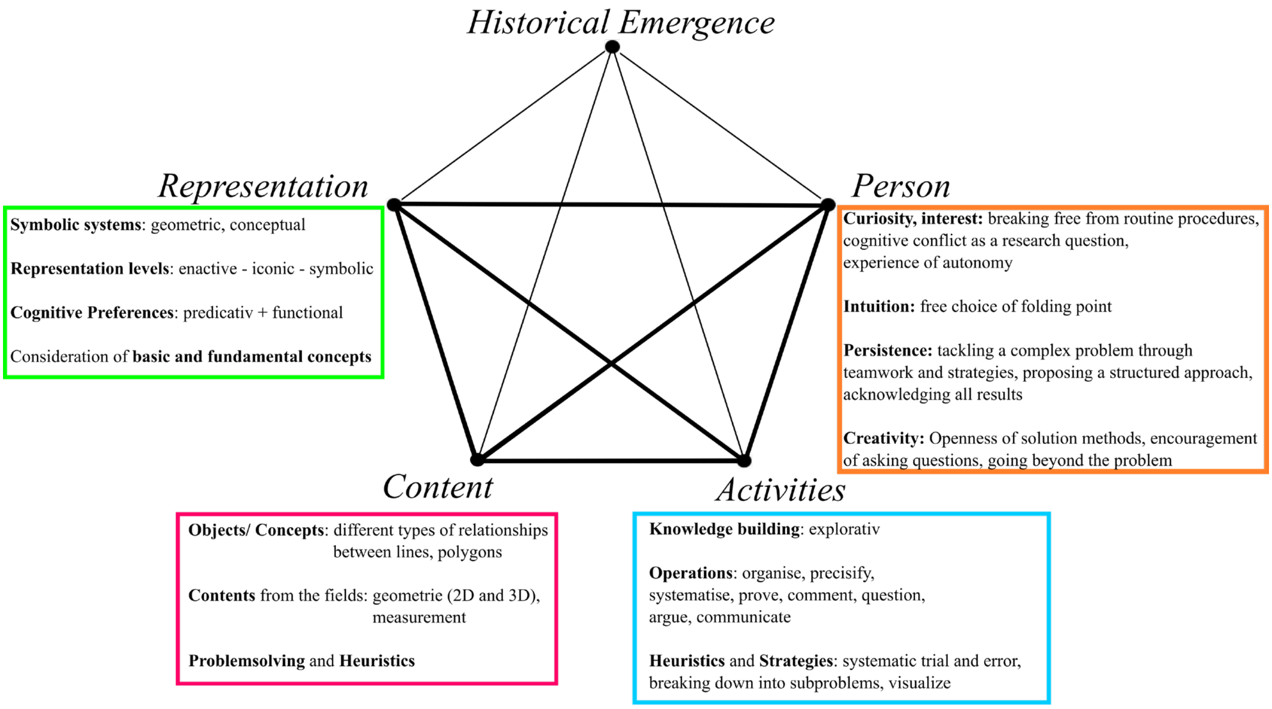
The resulting Pentagraphic Net can be used by teachers as a didactic tool for the descriptive analysis of networking possibilities present or omitted in textbooks. The nodes serve to capture the Fundamental Ideas addressed in the material, while the edges arise from the connections stimulated between nodes in the analysed material. In everyday teaching, the following steps have proven effective: Firstly, a rough scan of the entire topic in the textbook is conducted, focusing on the 'Content' node. Anchor points are established and noted down to identify the 'common thread' in a textbook chapter. The other nodes serve as points of orientation and may draw attention to specific features at this stage. This is followed by a detailed analysis with a focus on each individual node. Starting from the 'Content', the 'Activities' prompted in the textbook (types of knowledge construction, intra- and extra-mathematical activities, strategies, and heuristics) as well as the 'Representations' used (types of presentations, symbolic systems, cognitive preferences) can be examined. The historical 'Emergence' node (e.g., historical solution methods) and the 'Person' node (research questions that spark interest and curiosity, open and/or complex tasks that allow room for intuition and creativity while demanding persistence) then come into focus. This detailed analysis reveals the Fundamental Ideas contained in the textbook for each node, which then serve as a basis for discussing the edges (von der Bank, 2016).

The Pentagraphic Net can also be of use in a normative sense for lesson planning. Drawing from my practical teaching experience, I would like to share how this can be implemented. To illustrate, I will discuss the planning of a geometry lesson for 10-year-old students. Using the Pentagraphic Net for analysing the textbook revealed various content such as geometric shapes and bodies, surface areas, and concepts like parallel or distance. Thus, anchor points were set in the node 'Content'. Furthermore, it was noticeable that paper folding was frequently employed while simultaneously posing geometric problems (e.g., *Where are all points located that are equidistant from two given points?*). I decided to combine both approaches to generate excitement through a playful exploration using origami paper, while simultaneously providing room for the research process with a challenging problem.

The initial task for my students was as follows: *Mark a point anywhere on the square origami paper and fold the bottom left corner onto that point. Describe the shape of the folded surface.* While folding, the students were pleasantly surprised to discover that triangles and quadrilaterals could emerge. This immediately sparked curiosity and interest, leading to the research question: *Can we predict which point position leads to which figure?* As this is not an easy task, finding a solution relied on intuition and creativity, and it required persistence and individual support through teamwork. The result was a lesson with high student activity, focused work, and a lot of self-confidence when the solution was finally found at the end.

Figure 4 summarises my approach: I began my lesson planning with focus on the node 'Person' and the guiding question: What could be a problem that arouses curiosity and interest, while being challenging enough to require intuition, creativity, and perseverance for its solution? The posed problem demands mathematical 'Activities', such as questioning, organising, arguing, and essential strategies like visualising and breaking down into subproblems. The treated 'Content' includes geometric shapes and the concept of distance. This is crucial for the solution as all points resulting in triangles lie within two circular lines around the diagonal of the paper. Moreover, various 'Representations' were possible, organised according to enactive, iconic, and symbolic. Only the historical 'Emergence' remains unconsidered in this lesson. Naturally, geometric problems offer numerous opportunities to explore their historical development, but I deliberately deferred that to other lessons to avoid overburdening this introductory session.

Figure 4. Summarising my approach using the Pentagraphic Net.



5 Conclusion and Outlook

Mathematics, the practice of mathematics, and school learning encompass more than just cognitive aspects. This has been emphasised repeatedly. However, affect-related aspects such as joy, enthusiasm, curiosity, interest, intuition, creativity, and persistence are often overlooked in lesson design.

With the presented Pentagraphic Net, teachers have a manageable tool at their disposal that conveys the richness of mathematics in the classroom and stimulates the teacher's engagement with the subject matter. This enables them to make pedagogical decisions for their student group. At the same time, this approach to lesson planning acknowledges affect-related aspects as equally important as content, representations, activities, and historical emergence. Thus, a positive emotional experience in mathematics instruction can be emphasised from the planning stage onwards. Hopefully, the given demonstration of the practical use and the added value for lesson planning with the Pentagraphic Net, will encourage teachers to structure their teaching around Fundamental Ideas and, maybe, even start with a focus on Personality Ideas.

However, the role of the teacher in the classroom cannot be limited to planning lessons. Besides suitable learning materials, the demonstration of positive emotions related to the subject matter by the teacher is also a factor that can favour the learners' emotions (Frenzel & Götz, 2018, p. 116). Although empirical research yields contrasting results regarding the direct influence of positive emotions from both the teacher and the learners (Carmichael et al., 2017, p. 458), it is still worthwhile to reconsider the role of the teacher in the classroom.

The interplay between teacher and students cannot be described using the theoretical language of Fundamental Ideas. It resides in the atmosphere of the classroom and manifests through the demonstration of joy and enthusiasm for mathematics, as well as through a respectful and warm relationship between teachers and students. In this regard, educational psychology highlights three empirically researched personality traits of teachers that can help to create a positive learning environment. Specifically, they are as follows: *positive regard and attention, empathetic non-judgmental understanding of the students' world of experience,* and *authenticity* (Tausch, 2018).

It is easy to imagine that educators who possess these three personality traits would interact with their learners in a highly appreciative manner and create a conducive learning atmosphere in the classroom. This has also been supported by surveys conducted with learners. The students "were more spontaneous, expressed their own thoughts and feelings more openly in class, had less fear, perceived their teachers more positively (more respectful, empathetic, and genuine), and reported personal growth in the classroom" (Tausch, 2018, p. 639). The positive perception of the classroom by the students also has an impact on the feelings of the teachers. "In our studies, teachers' beliefs about their control over achieving instructional goals have been found to be significant for their personal experiences" (Frenzel & Götz, 2007, p. 294). This is also in line with Pekrun's Control-Value Theory, which suggests that we experience situations positively when we attribute a high personal value to them and simultaneously have a high sense of control over them (Pekrun, 2006).

So, a worthwhile next step would be to investigate how lesson planning with the Pentagraphic Net could also impact the classroom atmosphere. It would be interesting to explore if such planning leads to learning environments that are more positively perceived and valued by students. Furthermore, potential repercussions, in the sense of Pekrun's Theory, on teachers could then be investigated. In any case, a person-focused approach to mathematics instruction, in which both the emotional experiences of students and teachers are taken into account, offers a wide field of research.

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