Italian teachers' beliefs on covariational reasoning: an exploratory study

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Abstract: Covariational reasoning is the ability to grasp relationships between quantities and express them mathematically. Despite the recognized relevance of its understanding and knowledge, research shows that few students and teachers are able to adopt covariational reasoning. Our study aimed to investigate whether Italian teachers are aware of covariational reasoning and how and in what way it is present in Italian teaching practices. A teacher professional development course on the topic was launched, and questionnaires were administered at the beginning and end of the course. Teachers' views and beliefs about covariation and the possible changes regarding them after their participation in the course are investigated. The first analyses carried out on this data and initial reflections on the topic are shared.

Keywords: covariational reasoning, teachers' beliefs, meta-didactical transposition

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1 Covariational reasoning within the mathematics teaching and learning process

Covariational reasoning is the ability to grasp invariant relationships between quantities varying simultaneously (Thompson & Carlson, 2017). Its understanding and knowledge are important educational goals in mathematical and scientific learning at all school levels, from primary school onwards (Ferretti et al., 2024). Despite the internationally recognized relevance of covariation, research shows that only a small percentage of students and teachers is able to adopt covariational reasoning (Thompson et al., 2017). In the Italian educational paramount, covariational reasoning is largely absent from textbooks, and ministerial documents do not contain explicit references to this form of reasoning despite remarking on the relevance of the concept of function as a representation of real phenomena and the importance of mathematical modelling activities in teaching practices (Bagossi et al., 2022). These facts may be a possible reason for the absence of a covariational approach in school practices: most Italian teachers are not aware of covariation and its importance and, therefore, do not promote its use in their classes. In line with the literature (e.g., Thompson, 2013), these shortcomings become significant as teachers' understanding of covariation is a key factor in students' mathematical understanding of such a concept. This evidence of lack of awareness led us to design and implement a teachers' professional development (PD) course to overcome the shortcomings in the covariational reasoning field. The data collected throughout the PD course enabled us, as researchers, to investigate to what extent Italian teachers are aware of covariational reasoning (if they even refer to it correctly) and their beliefs related to this form of reasoning. The PD course, as detailed in the following paragraphs, was organized based on the Meta-Didactical Transposition theory (Arzarello et al., 2014) to facilitate the sharing and incorporation of covariational reasoning praxeologies within the teachers' community. Data gathered at the beginning and at the end of the PD course allowed for an exploration of the course's influence on teachers' awareness of and beliefs about covariational reasoning and its importance for students' understanding of functions.

2 The relevance of mathematics teachers' beliefs

Our study is placed among international studies dealing with how affective aspects – e.g., emotions, beliefs, attitudes, values – influence mathematics teaching-learning. This issue in literature has been extensively investigated, initially studying the relationship between affect and observable behaviours in students (e.g., Schoenfeld, 1989), and subsequently moving to study this relationship in teachers (e.g., Philipp, 2007). The relationship between beliefs and behaviour turned out to be even more problematic when researchers began to investigate teachers' viewpoint. Referring to the difference between *espoused beliefs* and *beliefs in practice* pointed out by Schoenfeld (1989), some studies raised contradictory positions about the consistency between teachers' espoused beliefs about mathematics teaching and their implemented didactic choices (e.g., Raymond, 1997). As the literature shows, although it is not appropriate to consider teachers' expressed beliefs as predictors of practice (Wilson & Cooney, 2002), it is now internationally recognised how teachers' beliefs are representative of their intentions of practice (Liljedahl, 2009). Researchers (e.g., Richardson, 1996) observed more generally that the relationship between affective aspects and observable behaviours cannot be described just as a direct cause-effect relationship, but it is necessary to account for greater complexity, due to their continuous and mutual influence. As stated by Schoenfeld (1989), collecting teachers' espoused beliefs remains a useful means for gathering information about the possible "playground" where teachers' didactic choices take place. Several studies have confirmed that it frequently occurs that pre- and in-service teachers have beliefs about both mathematics and its teaching that are not aligned with what educational research supports (Philipp, 2007; Richardson, 1996). Literature also reports examples of training paths focused on reflections and on changes in teachers' beliefs (e.g., Liljedahl, 2005). Grootenboer (2008), for example, stresses the need to create opportunities to conduct a review of the episodes from which beliefs originated and to have new experiences in which other types of beliefs can succeed. Despite several studies in this field, it is still unclear what impact teachers' PD courses actually have, and especially whether the change is gradual and quite slow (Ambrose, 2004) or more sudden in nature

(Liljedahl, 2010). According to Speer (2008), the most effective PD courses are characterised by a focus on meaningful aspects of practice and a recognition that teachers make sense of new information in light of their existing knowledge, beliefs, and practices. It is precisely in this line of thought that our study fits and on which the design and implementation of our PD course are based.

3 Theoretical framework

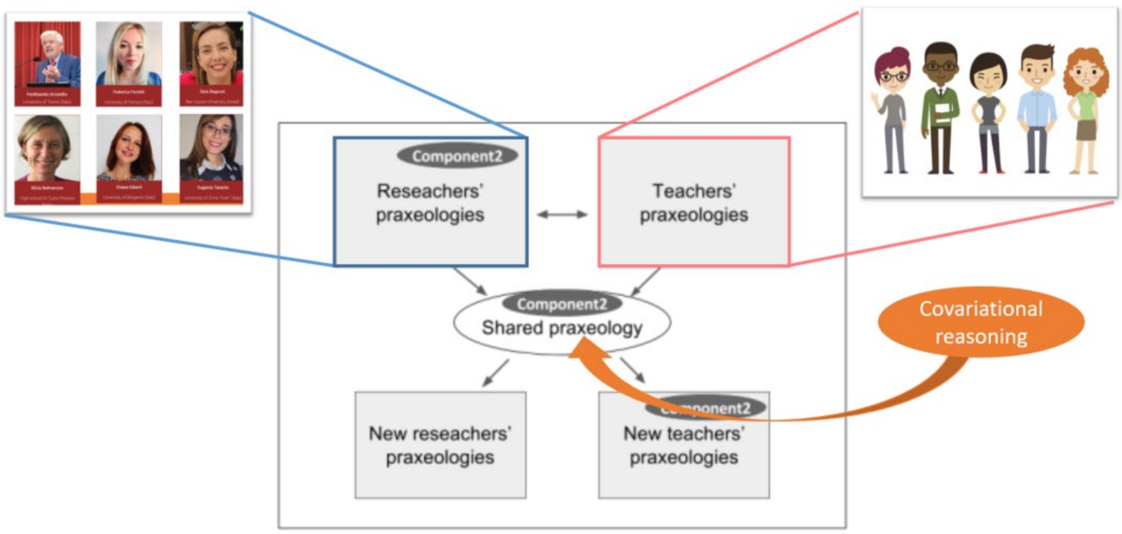
To describe and interpret some variables in our teacher's professional development course, we resort to Arzarello's et al. (2014) Meta-Didactical Transposition (MDT) theoretical framework. The MDT takes into consideration the practices of mathematics educators/researchers and those of teachers, the so-called *praxeologies*, when both communities are engaged in teachers' education activities. It is an adaptation of the Anthropological Theory of Didactics (Chevallard, 1992) to teacher education, through the integration of further elements. The Anthropological Theory of Didactics conceives the teaching activity as a didactical praxeology, which is made up of a set of tasks that drive the practice (praxis), the techniques that allow individuals to solve the problems, and the justification and theories (logos) that ground the techniques. Within the MDT, didactical praxeologies become meta-didactical praxeologies since they refer to the practices and reflections, which characterize teacher education processes. Meta-didactical praxeologies deal with practices and the theoretical reflections developed in teacher education activities. The components of a praxeology can be internal or external to a community (or an individual): internal if used by members of a community (or an individual), external if not. The aim of a PD course is to transform praxeological components that are initially external to the teacher community into internal ones (e.g., tasks and techniques around the use of technology for learning, theoretical findings from teaching research, ...). If a component of a praxeology moves from outside to inside, then the teacher community can evolve towards sharing this component among teachers and with researchers.
As we have already pointed out, covariational reasoning is often explicitly absent in teaching and learning processes, at least in Italy. So, it can be considered as a praxeological component initially external to the teacher community. Through our PD course, we aimed to extend the teachers' mathematical knowledge of covariational phenomena and make teachers aware of covariational aspects in activities they could already do in their classrooms. Therefore, in this study, we aim to investigate the course's influence on beliefs regarding covariational reasoning in order to understand whether the meta-didactical praxeologies shared in the course have become internal for the participating teachers.

4 Research context

The implemented teachers' PD course was focused on covariational reasoning and was addressed to Italian in-service mathematics teachers from all school levels. The course consisted of seven meetings of two hours each from November 2021 to January 2022 and a final meeting in April 2022 to report feedback from the experiments that the teachers had conducted in a later stage. The course, which involved a total of 41 teachers from various regions of Italy, took place online and synchronously. Its structure was designed in such a way that theoretical aspects alternated with practical examples provided by the researchers. Teachers worked in small groups divided by school level - first cycle (grade 1-8) and second cycle (grade 9-13). During the working group sessions, aimed at designing activities promoting students' covariational reasoning, the interaction and feedback between teachers and researchers were ongoing. Optionally, the designed activity could then be experimented, adapting it to each classroom context.

The main goal of the PD course was to make covariational reasoning evolve from an external into an internal, and possibly shared, praxeology (Figure 1)[1].

Figure 1. A schematic description of our PD course in light of the MDT lens (adapted from Taranto et al., 2020, p. 1441)



In light of the clues offered by the PD course, the meta-didactical praxeology that we, as researchers, intended to transpose to teachers in training was set up as follows:

- Task: designing a teaching activity that stimulates covariational reasoning in students;
- Technique: design to be done in groups, with an invitation to experiment (taking into account refinement on one's own classroom context);
- Justification: peer work, encouraging collaboration and discussion also with the course educators;
- Theory: awareness that teacher espoused beliefs might not coincide with the beliefs in practice if there is not an impact on teachers' teaching practices.

[1] Note. The community of the researchers - all members of the international mathematics education MAT&L research group - was formed by the authors of this contribution, jointly with the researcher-teacher Silvia Beltramino from High school Maria Curie (Pinerolo) and Chiara Giberti from the University of Bergamo.

5 Methodology

To investigate the teachers' beliefs on covariation before starting the course and what impact the PD course had on them, two questionnaires were administered, one at the beginning (IQ) and one at the end of the course (FQ), with some questions in common. Both questionnaires consisted of multiple-choice, open-ended, and graduated scale questions. In the following, we will focus only on some of these questions and in particular on those that the two questionnaires had in common (5 open-ended and 2 graduated scale). We will denote questions with IQ.x or FQ.x (for x ranging from 1 to 14). Of the questions related to the graduated scales, we will focus only on some of their items and specifically some of those related to the perceived relevance of covariation in learning processes in mathematics (from 1=strongly disagree to 5=strongly agree).
The questions selected for this preliminary analysis were also chosen according to the aspects they address concerning knowledge/beliefs about covariational reasoning (i), about teaching covariational reasoning (ii), and about students' learning of covariational reasoning (iii) (Table 1). Furthermore, such selected questions from the initial and final questionnaires help us to trace the transposition of this praxeology on teachers and its evolution.

Among the 41 teachers who completed the PD course, 39 of them filled in both IQ and FQ. We underline that in the IQ, after IQ.3, the teachers were provided with a definition of covariational reasoning (e.g., Thompson & Carlson, 2017) and in light of that and their previous knowledge, they were asked to answer. The analysis process of the open-ended questions (IQ.14, FQ.1, FQ.2) was structured in open coding for the generation of the categories. Then, the classifications and identification of the themes were carried out by the authors of the study, reaching a good degree of agreement. The analysis of the graduated items takes into account frequencies, means (M), and standard deviations (SD).

Table 1. Questions selected from IQ and FQ

	i	ii	iii
IQ.2: At which level of school do you teach?			
IQ.3: Have you ever heard of covariation among quantities?	x		
IQ.14: Now, how would you define covariation in your own words according to your professional experience?		x	
FQ.1: At the end of this course, how would you define covariation in your own words?	x		
FQ.2: Which of the ideas you initially had about covariation were confirmed or refuted during the course? Tell.	x	x	x
IQ.10 & FQ.6: How much do you agree with the following statements? [**Item 6.** Students can understand functions without maintaining covariational reasoning.]			x
IQ.10 & FQ.6: How much do you agree with the following statements? [**Item 8.** Sustained attention to students' covariational reasoning is necessary from early grades on.]		x	x

6 Results

Among the 39 teachers (Ti) who answered both questionnaires 12 were teachers in the first cycle and 27 in the second cycle (IQ.2). 34 of them declared that they had heard of covariation (IQ.3) before the PD course, but then the answers to open questions revealed a variety of beliefs about this concept.

The answers to question IQ.14 were classified into the categories in Table 2.

Table 2. IQ.14 [Now, how would you define covariation in your own words according to your professional experience?] and FQ.1 [At the end of this course, how would you define covariation in your own words?]

Category	**IQ.14**	**FQ.1**
Link of variation between quantities	15	29
Misunderstanding with covariance	9	2
Relations between quantities in a specific domain	5	2
Generalisation of a law with parameters	1	/
Generalisation of a law	/	4
Form of reasoning	/	1
I do not know	5	0
NA (not available)	4	1
Total	**39**	**39**

Most of the respondents (15 out of 39) speak of covariation as a variation link between quantities. For example, T11 writes: “The mode of variation of one quantity in relation to the variation of another quantity”. We also observe that 9 out of 39 teachers confuse covariation with statistic covariance. T16 writes: “Covariation is an indicator of how two or more variables are related”. Five out of 39 believe that covariation concerns variables related to specific domains (“Establishing relationships between changes in related quantities mainly in the economic sphere [...]” [T7]). Another 5 out of 39 admit that they do not know what covariation is and 4 others do not answer at all.

When asked the same question at the end of the course, through question FQ.1, we observe that the categories generated are very similar to the previous ones (Table 2), but the distribution of frequencies changes markedly. There are now 29 out of 39 teachers who speak of covariation as a variation link between quantities. For example, T5 writes: “I would define it as the relationship between two pieces of data taken at different times of a changing situation.”. Misunderstanding with covariation remains for only 2 out of

39 teachers, just as the number of teachers who believe that covariation only affects specific domains drops to 2. Now, there are no longer any teachers who do not know what covariation is, and there is only 1 who does not answer the question.

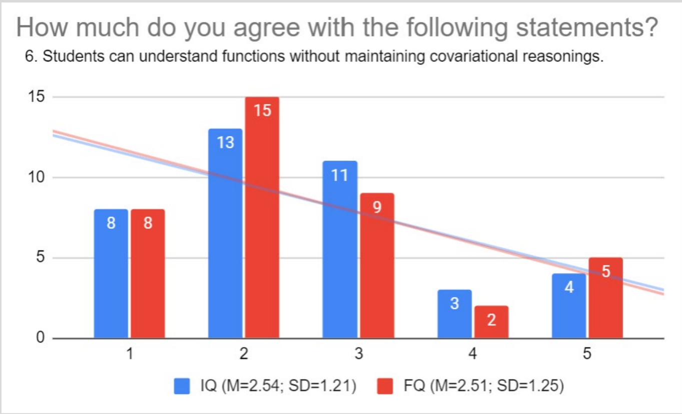
Question QF.2 allows us to better delineate what evolutions took place during the course, because it asks the trainees to reflect on which initial ideas they had changed or were confirmed during the course. Table 3 shows that 16 out of 39 respondents had confirmation of what they already knew on the subject. On the other hand, 21 out of 39 had denials: 8 admitted that they were confused with covariance and 9 that they understood what was meant by covariation.

Table 3. QF.2 [Which of the ideas you initially had about covariation were confirmed or refuted during the course? Tell.]

CONFIRMS		**DISCLAIMERS**	
Importance of using graphs	4	Consider the situation and not just the data	1
Importance of introducing covariation at school	3	Confusion with covariance	8
Confirmation of own ideas and enrichment	8	Covariation is complicated for students	3
Variables and parameters	1	Clarification of what covariation is	9
TOT	**16**		**21**
NA (not available)	4		

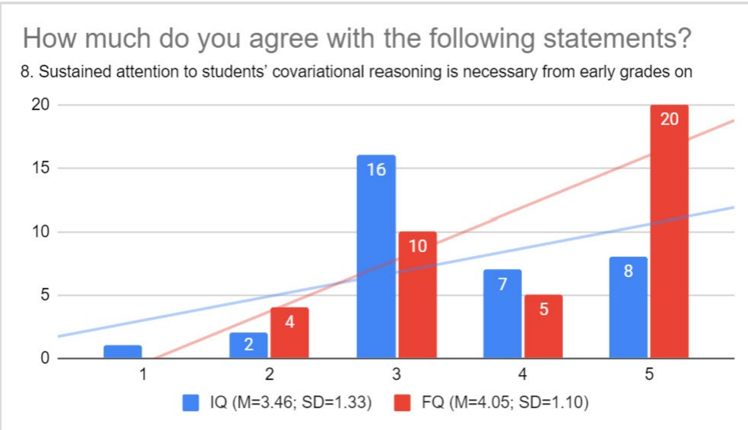
Concerning the selected graduated scale questions, administered in both the IQ and the FQ, we observe the trend of item 6 "Students can understand functions without maintaining covariational reasoning". From the graph in Figure 2, in general, it would seem that the frequencies have not undergone particular changes between the before and after the course. However, a more detailed analysis, which we will not go into in this paper, shows that in the FQ the answers of individual teachers are not confirmations of their previous answers. So, this denotes a change in the way teachers see the concept of function. Indeed, these teachers have realised that covariational reasoning is important for a less static approach to functions.

Figure 2. Answers to item 6 both in IQ and FQ



A similar trend occurs for item 8 "Sustained attention to students' covariational reasoning is necessary from early grades on". Here, the frequency distribution between before and after the course is somewhat more pronounced (Figure 3). This denotes a change in the way teachers perceive the importance of the concept of covariation already at early school levels.

Figure 3. Answers to item 8 both in IQ and FQ



7 Final remarks

As we can see from the analysis of the results of the IQ, compared to the photograph of the moment analysed, it seems that in the Italian panorama there are neither a solid knowledge of covariational reasoning nor beliefs in line with the literature regarding the importance of covariational reasoning in the processes of learning and teaching mathematics.

In detail, from the data analysed, we can observe that the covariational reasoning, at first external to most teachers, has become a praxeology shared by both researchers' and teachers' communities. In fact, whereas in the IQ some teachers had no idea how to define covariation or confused it with covariance, in the FQ they all show that they not only understand what it is, but also display a more appropriate use of terminology in explaining it. We also see confirmation or denial of what were their initial conceptions about covariation and how it could be presented to students. The illustrated data also show how teachers have reflected on the potential of covariation in approaching functions in a less static way and how important it is to propose covariational reasoning as early as primary school. These reflections are in line with studies on covariational reasoning in mathematics learning and teaching processes (Ferretti et al., 2024; Thompson & Carlson, 2017,).

Although the one proposed here is a preliminary analysis, the data illustrated allows a perception of changes in beliefs and views on covariational reasoning. On the one hand, more knowledge/beliefs about covariational reasoning emerge (e.g., comparing IQ.3 with FQ.1), thanks to the theoretical knowledge that the PD course offered. On the other hand, the perception of change is also an expression of greater knowledge/beliefs about teaching covariational reasoning (e.g., comparing IQ.14 with FQ.2) and about students' learning of covariational reasoning (e.g., comparing IQ.10 with FQ.6, items 6 and 8). These aspects are to be considered linked to the examples of activities expendable in the classroom that the PD course offered. They are also connected to the experiments that some teachers conducted during the PD course, and whose results were always shared and reflected upon during the PD course.

Although these changes are significant to make a training course effective (Speer, 2008), we certainly cannot say that all teachers have benefited equally from this development in their praxeologies. However, we can speak of a perceived change in the teachers' praxeologies as a result of an internalisation of the meta-didactical praxeology that the training course sought to transpose. The detected glimpse of changing beliefs, which is an essential component for teachers' practices change, will underpin future, deeper investigations.

Research ethics

Author contributions

This paper is the result of equal contribution from all the authors involved, reflecting a collaborative effort in both research and writing. All authors have read and agreed to the published version of the manuscript.

Informed consent statement

Informed consent was obtained from all research participants.

Data availability statement

The data that support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

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