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# Editorial

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This Special Issue in International Journal on Math, Science and Technology Education (LUMAT) has a unique history. It is a collection of selected papers from the 27th International Conference on Mathematical Views (MAVI27), which took place in Bremen, Germany, from 15 to 17 September 2021. At least that was the intention!

The unique story begins two years earlier, when Maike Vollstedt and her team (at that time Christoph Duchhardt, Neruja Suriakumaran, and Aylin Thomaneck, supported by Kerstin Düren, Vesife Hatisaru, and Ralf Erens) agreed to organize MAVI26 in Bremen in August 2020. Everything was already planned when the COVID-19 pandemic took over the world, making it impossible for participants to travel to Bremen. As an alternative, the organizers switched to an online conference format via Zoom. The opening plenary was given by Andreas Eichler, Federica Ferretti, and Andrea Maffia, who presented „A comparative study on German and Italian prospective teachers' view on mathematics", that originated during MAVI25 in Intra (Italy), in 2019. The closing plenary was given by Markku S. Hannula on „Revisiting the metatheory of affect: Special focus on studying states".

As the conference had already been completely organized locally, the Bremen team (then Maike Vollstedt, Martin Ohrndorf and Aylin Thomaneck, supported by Kerstin Düren, Vesife Hatisaru, and Ralf Erens) agreed to also host MAVI27. Unsurprisingly, the pandemic threw over everything again. The number of cases worldwide was on the rise, making it unsafe to hold the conference on site. Similarly, in 2021, there was no alternative but to hold MAVI27 online again. The keynote speech was given by Stanislaw Schukajlow-Wasjutinski on the topic „What objects are you targeting? Strategy-based motivation and emotions".

Albeit their online format, both conferences provided a wonderful opportunity to engage in scientific exchange on affect-related research projects and to discuss the studies constructively and critically at a high level. The submissions underwent a peer review process, where every paper was reviewed by two other conference participants. As the MAVI community values a constructive and critical atmosphere, the review was unblind. Of the 15 submissions, 13 were accepted and presented at the conference.

These 13 papers were revised according to the review reports and submitted for publication in the proceedings, following the conference. During the publication process, they underwent an additional review where every paper was reviewed by two reviewers. One reviewer was usually the reviewer of the pre-conference review, and the second reviewer was someone who was not previously engaged in the review of the respective submission. After this robust review process, we are pleased to bring together the following nine selected papers in this Special Issue.

The Special Issue brings research papers together on affect in mathematics education across all school levels and beyond. Topics of the papers on the focus include beliefs, attitudes, emotions, interest, motivation, identity, mindset, and interpersonal relationships. In 'Investigating the Complex Relations Among Affective Variables in the Context of Gambling', Chiara Andrà, Eleonora Averna, Ilaria Copelli, Gianluca Sini Cosmi, Elisa Paterno, and Claudia Chiavarino touch upon a dramatics phenomenon that has been spreading in the context of Italy and beyond: gambling disorder. The authors investigated the role of affect in a sample of secondary students' gambling behavior to understand how activities in mathematics lessons can help to prevent students from this behavior. In 'Attitudes in Mathematical Discovery Processes: The Case of Alex and Milo', Carolin Danzer presents six-grade students' attitudes in mathematical discovery processes and how they handle with counterexamples. As a modification of subject matter didactics, in 'Beliefs-oriented Subject-Matter Didactics – Design of a Seminar and a Book on Calculus Education', Frederik Dilling, Gero Stoffels, and Ingo Witzke focus on the students' views of a specific mathematical content: Calculus. As part of a larger research, Andreas Ebbelind and Tracy Helliwell present the language of one mathematics educator by utilizing the Systemic Functional Linguistics framework in their study 'Examining Interpersonal Aspects of a Mathematics Teacher Educator Lecture'. In 'Emotional Classroom Climate from a Psychological Perspective', Ana Kuzle presents a study on how grade three and grade six students perceive the emotional aspect of their geometry lessons through participant produced drawings. Maria Kirstine Østergaard provides the findings of a systematic literature review on students' beliefs about mathematics as a discipline in her study 'Characterizing Students' Beliefs about Mathematics as a Discipline'. In 'A Quantitative Study about Describing Correlations of Motivational and Affective Aspects and Digital Heart Rate Measurement', Felicitas Pielsticker and Magnus Reifenrath present a unique study on motivational heart rate measurement of students in a workshop on graph theory. Anna Schreck, Jana Groß-Ophoff, and Benjamin Rott focus on associative and evaluative judgements of university students on mathematical epistemological beliefs (in other words, connotative aspects of epistemological beliefs) in their study 'Connotative Aspects of Epistemological Beliefs: A Pseudo-longitudinal Study with Students of Different Mathematical Programmes of Study'. In 'How to Deal with and Utilize (Mathematics (Education)) Researchers' Beliefs', Gero Stoffels concentrates on researchers' beliefs, a group whose beliefs are relatively less investigated.

The publication process took a while for a few reasons. First, we were still in a pandemic, which added to the workload, combined with difficult childcare situations and the like. Second, the organizing committee hosted the conferences free of charge, so there was no fund to spend on publication of the proceedings. Several options were considered. The Editorial Team was unwilling to make an agreement with any large international publisher, as this would have meant publishing with a charge; the papers could have been accessed with cost limiting their read, and citation accordingly. Another option was considered to publish the proceeding open access supported by the University of Bremen. Finally, the MAVI Board, in collaboration with Markku S. Hannula, worked out the possibility of publishing this Special Issue in LUMAT.

We are delighted to publish the proceedings in LUMAT, a high-quality international journal, and wish to thank Markku S. Hannula and the members of the MAVI Board for their wise and ongoing commitments to MAVI. Having said all this unique journey, we wish readers will gain rich insights into current studies on affective theories in the field of mathematics education upon viewing the papers available in the Special Issue.

*Yours in research,*
*Maike Vollstedt, Vesife Hatisaru, Martin Ohrndorf, and Aylin Thomaneck*
*Editorial Team*

# Beliefs-oriented subject-matter didactics: Design of a seminar and a book on calculus education

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This paper presents a modified approach to subject-matter didactics, in which the focus is not on the content itself, but on the students' view of the content. The introduction deals with an overview of subject-matter didactics and the notion of beliefs used in this paper. The main portion of the paper deals with presenting the concepts of a book and a seminar based on the student-centered subject-matter didactics approach. For the first qualitative evaluation, selected reflections of students are analyzed. Finally, initial findings are summarized and an outlook is provided.

Keywords: belief systems, calculus education, pre-service teacher training, subject-matter didactics

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## 1 Introduction

The analysis of mathematical content and contexts as well as their translation into practical concepts for teaching mathematics at school are central tasks of mathematics education research. In German-speaking countries, there is a long tradition in this field of research under the title "Stoffdidaktik" or "Subject-Matter Didactics" (cf. Hefendehl-Hebeker, 2016), which still has a strong influence on research and teacher education today:

> Stoffdidaktik has been a dominant approach to mathematics education research within the German speaking countries, which puts the analysis of the mathematical subject matter at its heart. It has been the prominent approach to research until the 1980s. Nowadays, it still influences research in mathematics education in German speaking countries. (Hußmann et al., 2016, p. 1-2)

The focus of classical subject-matter didactics is the mathematical content taught at school. The aim is to provide students and teachers with an accessible approach to mathematical content knowledge. For this purpose, the subject-matter didactics investigate:

- “Essential concepts, procedures and relationships including appropriate formulations, illustrations and arrangements for teaching
- Essential structures and domain-specific ways of thinking

- The inner network of paths by which the components are connected and possible learning paths throughout the domain" (Hefendehl-Hebeker et al., 2019, p. 26).

In this paper, a slightly modified and novel approach to subject-matter didactics is described and applied. Instead of assuming a fixed mathematical framework, which is "elementarized" for school, the focus lies on different ways of thinking about mathematical concepts, disciplines, and mathematics in general. Theoretical approaches and empirical findings in the context of mathematics-related beliefs form the basis of learner-centered subject-matter didactics as will be described in the following sections, using calculus education as an example. The authors present the design of a seminar and a book on calculus education using this approach. Furthermore, a brief insight into the evaluation of the seminar and the book at the University of Siegen in the summer of 2020 is provided.

## 2 The underlying notion of beliefs

The general idea for the conceptualization of the textbook and the seminar on calculus education lies in the concept of belief and its application in calculus. There are many definitions of the term "belief" (cf. Thompson, 1992) and related terms (cf. Pajares, 1992), which differ considerably. In our conceptualization, we rely on the well-known definition of Schoenfeld (1985), who considered beliefs as mental structures that determine the behavior of a person:

> Belief systems are one's mathematical world view, the perspective with which one approaches mathematics and mathematical tasks. One's beliefs about mathematics can determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. Beliefs establish the context within which resources, heuristics and control operate. (Schoenfeld, 1985, p. 45)

Hence, Schoenfeld (1985) defined the term "belief system" in relation to a person's behavior when dealing with mathematical problems. In particular, Schoenfeld applied the term to the description of problem-solving situations. However, the concept can be applied to mathematical knowledge development processes in general. According to Schoenfeld, beliefs and belief systems are mental structures with cognitive and affective components (cf. Schoenfeld, 1985, 1992), which substantially determine behavior in addition to other important factors (resources, heuristics, and control).

The decision to make mathematics-related beliefs the basis of a subject-matter-oriented seminar and a book can be justified by the idea that those beliefs have a considerable impact on students' learning of mathematics. This idea is also described in the well-known quote by Goldin et al. (2009):

> To sum up, *beliefs matter*. Their influence ranges from the individual mathematical learner and problem solver and the classroom teacher, to the success or failure of massive curricular reform efforts across entire countries. (p. 14, emphasis in the original)

The development of adequate beliefs about mathematics by prospective teachers can be understood as a critical goal of the teacher-training program, subsequently affecting teaching at school and, in turn, also the development of beliefs by students in mathematics classes:

> Because attitudes are acquired in learning processes in which the (social) environmental conditions have a substantial influence, it can also be argued that the attitudes of teachers have a substantial influence on the attitudes of students — on one hand, in direct communication and interaction in a mathematics class, and on the other hand, indirectly through the concrete design (choice of material and methods, and assessment system) of a mathematics class. (Grigutsch et al., 1998, p. 4, authors‘ translation)

## 3 Beliefs for enabling perspectives on calculus

In this work, we are particularly interested in domain-specific beliefs (i.e., those that refer to a specific mathematical domain — in our case, calculus). The basis for our work can be traced back to the article "Domain-Specific Beliefs of School Calculus" by Witzke and Spies (2016). In this article, the authors discussed, among other things, that some domain-specific beliefs in school calculus are more dominant than others, according to the way calculus is taught at school and the way students receive and construct the knowledge. Based on a qualitative content analysis with the inductive specification of deductive categories, Witzke and Spies (2016) identified six deductive categories (for examples of the categories from the data, see Witzke & Spies, 2016, p. 144):

- *Logical-structural orientation*, which focuses on deduction and proof, as well as the understanding of (intra-mathematical) connections among concepts and their underlying structures.

- *Abstract-terminological orientation*, which focuses on formal rigor, the use of precise mathematical language, and the understanding of mathematical objects as abstract entities.
- *Toolbox orientation*, where operating calculus performs certain rules, formulas, and procedures in a schematic way (e.g., how to determine a derivative, extreme values, or similar).
- *Utility orientation*, which focuses on extra-mathematical applications or mathematical modeling.
- *Empirical orientation*, in which objects related to the real world and basic concepts derived from these perceptions are the focus.
- *Symbolical orientation*, in which objects of calculus are identified with characteristic symbols.

## 4 Structure of the seminar and the book

Various German books on calculus education (German: Didaktik der Analysis) in the tradition of subject-matter didactics are available. Classics include, for example, "Analysis verständlich unterrichten" (English: Teaching Calculus in a Comprehensible Way) by Rainer Danckwerts and Dankwart Vogel (see Danckwerts & Vogler, 2006) or "Didaktik der Analysis" by Werner Blum and Günter Törner (see Blum & Törner, 1983). However, there are also more recent books such as "Didaktik der Analysis: Aspekte und Grundvorstellungen zentraler Begriffe" (English: Calculus Education: Aspects and Basic Ideas of Essential Concepts) by Gilbert Greefrath, Reinhard Oldenburg, Hans-Steffan Siller, Volker Ulm, and Hans-Georg Weigand (see Greefrath et al., 2016).

All these books take different perspectives on calculus and the teaching of calculus (e.g., a focus on extra-mathematical application). Generally, their structures follow the systematic structure of calculus (i.e., start with basic concepts such as sequences and series, and afterward discuss functions, derivatives, and integrals, as well as their adequate introduction depending on the authors' perspectives). The seminar and the textbook of calculus education presented in this paper take a different approach. The content is structured in terms of typical belief systems of mathematics according to different educational perspectives on calculus, namely, the formal-abstract perspective, the empirical-concrete perspective, the "toolbox" perspective, and the application perspective. These four perspectives are the result of a reduction of the orientations mentioned in the study by Witzke and Spies (2016) — the logical-structural orientation and the abstract-terminological orientation merge into the formal-abstract perspective, and the toolbox orientation and the symbolic orientation form the “toolbox” perspective.

This is a subjective selection from the perspective of the authors of this paper, based on their experience with calculus teaching. The decision to use the term “perspectives” instead of “beliefs” or “orientations” in the structure of the textbook is due to the fact that perspectives are actively taken, whereas beliefs or orientations do not necessarily become explicit for the belief-bearers, but possibly represent a "hidden variable" (Goldin et al., 2009) for them. This reinterpretation enables beliefs to be addressed proactively and productively for mathematical teaching and learning. Each of the four perspectives forms a chapter of the textbook or a unit of the seminar, which connects the basic mathematical concepts of calculus and interprets their educational implications according to the focused perspective. In addition, a concluding chapter interconnects the different perspectives on calculus according to a higher point of view gained in the course of the book and the seminar.

The aim is to experience that there is a multitude of views on mathematics or perspectives on mathematics in general and on calculus in particular. The higher point of view is characterized by the knowledge of the manifold of these perspectives and the ability to address adequate belief systems in different contexts of application (a school, university, teacher, or learner) or to adopt and further develop them. An argument for such a conception, under the condition that the students are already familiar with calculus, is offered by the following quotation from "Elementarmathematik vom höheren Standpunkt" (English: Elementary Mathematics from a Higher Standpoint) by Felix Klein:

> I shall by no means address myself to beginners, but I shall take for granted that you are all acquainted with the main features of the most important disciplines of mathematics. I shall often have to talk of problems of algebra, of number theory, of function theory, etc., without being able to go into details. You must, therefore, be moderately familiar with these fields, in order to follow me. [...] In this way I hope to make it easier for you to acquire that ability which I look upon as the real goal of your academic study: the ability to draw (in ample measure) from the great body of knowledge taught to you here as vivid stimuli for your teaching. (Klein, 2016, p. 1f.)

Hence, the book and the seminar are based on the assumption that a major goal of teaching calculus should be the stimulation of multiple perspectives (see also Green, 1971 on the formation of beliefs as a goal of mathematics education), whereas the formal-abstract perspective does not necessarily have to be taken by students at school. Research on the transition from school to university in mathematics shows that the change of understanding from an empirical-concrete to a formal-abstract belief system of mathematics is associated with major challenges on an epistemological level (cf. Stoffels, 2020; Tall, 2013). To prepare students for university, teachers should nevertheless be aware of this perspective so that teaching does not obstruct this change in the belief system. This can be achieved, for example, by focusing on mathematical activities that are characteristic of the empirical-concrete as well as the formal-abstract belief systems, such as deductive reasoning or the use of symbolic calculations, which are independent of whether mathematics is understood as an ontologically bound empirical discipline or as an abstract formalistic science.

As an introduction to the topic, the first chapter of the book or the first unit of the seminar explicitly addresses the concept of belief on the basis of varied research literature (i.e., Grigutsch et al., 1998; Schoenfeld, 1985) and illustrates it with original examples. To approach one's own belief system of "school calculus" and to reflect on it while working with the book or in the seminar, several stimulating questions are provided:

1. Why should calculus be taught in school?
2. What are typical activities that you associate with calculus at school?
3. What are typical topics that you associate with calculus at school?
4. When do you consider a statement in calculus to be verified?
5. Why should you, as a prospective mathematics teacher, attend a lecture on calculus during your studies for a teaching profession?
6. What are typical mathematical activities that you associate with calculus at university?
7. What are typical topics that you associate with calculus at university?

Finally, the first chapter also presents the normative goals of teaching calculus with reference to German curricula (Conference of the German Ministers of Education and Cultural Affairs, 2015) and the domain-specific educational literature (e.g., Dankwerts & Vogler, 2006; Greefrath et al., 2016).

In the subsequent four chapters, the formal-abstract perspective, the empirical-concrete perspective, the "toolbox" perspective, and the application perspective are initially discussed separately. In this context, the following topics (and others) are addressed:

- Formal-Abstract Perspective:

  - Formal elements in school calculus
  - Real numbers and functions
  - Central theorems of calculus
  - Theorems of calculus at school
- Empirical-Concrete Perspective:
  - Visual representations in the teaching of calculus
  - Empirical belief systems in the history of calculus
  - Geometric representations of theorems in calculus
  - Illustrative approaches to digital technologies
  - Tools and tactile models in calculus teaching
- "Toolbox" Perspective:
  - Algorithms in calculus education
  - Extreme values and other characteristics ("Funktionsuntersuchung")
  - Determination of functions
  - Determination of extreme values
  - Rules of derivation and integration
- Application Perspective:
  - Interdisciplinary mathematics teaching
  - Modeling with functions
  - Applications from the natural sciences
  - Applications from economics

Certainly, by limiting the sections to one perspective, one loses the rich linking of perspectives. In school calculus, such changes of perspective should also be attempted, as long as they are consciously and intentionally stimulated by the teacher or reflected by the students. To support the multi-perspective view of the students, who are expected to have already experienced the interconnection of the different perspectives in the context of their own calculus education, the authors consciously decided to initially present separate perspectives on calculus — and present them in an integrated way on a meta-level in the last chapter. Thus, in the final chapter, findings on concept development in the context of calculus are discussed. These findings refer to the concept of "Grundvorstellungen" (Vom Hofe & Blum, 2016) as well as subjective domains of experience (SDE) (cf. Bauersfeld, 1983) to describe overarching and interconnecting perspectives on school calculus.

## 5 Evaluation of the seminar and the book

The book described in this article is based on a script for a seminar on calculus education by Frederik Dilling and Ingo Witzke from 2018. They tested for the first time at the University of Siegen the approach of using belief systems as a basis for studying calculus. In the following three semesters, Frederik Dilling and Ingo Witzke together with Gero Stoffels conducted the seminar with this concept again. The experience from these four semesters was used to further develop the book as well as the underlying concept and to adapt it to the requirements of university students as the main audience.

Three of the semesters took place during the COVID pandemic, so the courses were arranged in a distance-learning format. For this reason, a reading course was conducted with a two-week interval for reading one chapter of the book, completing selected exercises from this chapter, and finally obtaining written feedback from the lecturers. In addition, the topics of the chapters were discussed in groups through videoconferences on several dates.

The following brief insight into the evaluation of the course and the book is based on the summer 2020 semester — the second time the course was conducted. In total, 10 bachelor's students of teaching mathematics for "Gymnasium" (high school) participated in the course. The authors of this paper conducted a hermeneutic descriptive analysis of selected answers to the exercises in the book as well as detailed written feedback on the content and structure of the chapters provided by the students. In this article, only a small glimpse into the data can be given as this is not a comprehensive empirical study.

In the first chapter of the book, the students were asked seven reflective questions to reflect on their own beliefs about calculus (see above). Among other things, they were asked to consider why calculus should be taught at school and university as well as what typical activities and topics they associate with calculus at school and university. Overall, the answers of the participating students to the question of why calculus should be taught at school mostly referred to the applications of calculus in daily life, future work, or as a prerequisite for university studies with a special focus on STEM disciplines. Only one student focused solely on the "foundational aspect of calculus for mathematics in general," "interconnections between mathematical fields," and fostering "abstract concepts and logical argumentations." The beliefs of the students on why it is necessary for them to learn calculus at university focused on deepening their understanding of calculus and their hopes for an improvement in their future teaching. Typical activities and topics assigned to school calculus were related to the "toolbox" perspective. By contrast, the mentioned university activities and topics can be assigned to the formal-abstract perspective. One student, for example, provided the following reflection on university activities:

> This is closely linked to the topics [of university mathematics]. The 'sitting on problems for a long time' but also talking with others about the exercise sheets. A lot of thinking, not 'understanding the principle and then using a calculator' as in school. Proving that it is also a big topic that is usually completely new to you. (authors' translation)

After this reflection, the students had to work on the well-known isoperimetric problem of maximizing the area of a rectangle to a given fixed circumference. After solving the problem, the students had to reflect if their answers to the previous reflective questions were fit for this task and their solutions. Hence, from the beginning of the book, the authors foster an awareness of the students' own beliefs and valuations. To illustrate this connection, the answer of another student is given, but it is important to mention, that at this moment, the student had a different notion of the formal-abstract perspective than that intended in the book:

> In the first task [these are the initial reflective questions], I wrote that a pupil should learn problem-solving strategies at school and be enabled to relate models to the environment. Hence, in my opinion, the empirical-concrete perspective and the formal-abstract perspective apply and are directly related to the initial task. (authors' translation)

Similar reflecting questions with a focus on the students' beliefs are provided throughout the book (e.g., in the chapter about the formal-abstract perspective, there are questions about the difference between calculus at school and calculus at university, as well as the significance of the concepts of continuity, differentiation, and integration). The key reflecting questions related to the final reflection on perspectives on calculus comprise the last exercise in the book:

> Take your time and reflect on your domain-specific beliefs of calculus and the teaching of calculus. To what extent were aspects of this didactics of calculus new to you? Which aspects do you want to pay special attention to for your future calculus lessons? (authors' translation)

All participants contributed a detailed reflection and were able to differentiate between the perspectives as well as to connect them, referring to the integrated framework of linking domain-specific beliefs with the SDE concept. The following response from a student is prototypical of the responses of the participants:

> Some aspects that were addressed here in the didactics of calculus were, perhaps, known to some extent (such as the different perspectives at school and university), so you had an idea, but it was very enriching to really see how different the approaches were. You might have experienced the consequences yourself. Some aspects such as the toolbox perspective were very familiar from high school [“Oberstufe”], especially those of the “Kurvendiskussion” [explanation: an examination of a function graph by calculating extreme values]. Higher-level constructs such as the SDE or “Grundvorstellungen” were new and offered a good opportunity to reflect for oneself on how one wants to later approach teaching on various topics. It has become important for me to link SDE and to offer the students a good structure that makes it easier to understand complex topics on the basis of areas that have already been covered. (authors’ translation)

## 6 Summary and outlook

The aim of this paper was to present a new approach to subject-matter didactics and to explain it using the context of calculus. For this purpose, the designs of a book and a seminar were presented. The short glimpse into the reflections of some students demonstrated that it is worthwhile to address different perspectives on the concepts, activities, and theorems and to make the differences explicit to encourage a deeper understanding and reflective perspectives on calculus. This can also mean that it might be advantageous to modify standard approaches for teaching calculus. For instance, this opens possibilities to not just strictly follow the systematic structure of calculus. However, the corresponding changes also create new challenges. For example, the systematic structure of the concepts in calculus moves into the background and the students have less awareness of it. The book described in this paper is scheduled for publication in 2023.

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# Connotative aspects of epistemological beliefs: A pseudo-longitudinal study with students of different mathematical Programmes of Study

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Various studies have shown that epistemological beliefs affect personal learning and teaching performances. Therefore, epistemological beliefs have become an attractive object of research with different methods of survey. A distinction can be made between denotative and connotative aspects of beliefs, the former being reflected upon, explicit beliefs, whereas the latter being associative and evaluative judgements on (in our case: mathematical) epistemological beliefs. The present study used the instrument Connotative Aspects of Epistemological Beliefs by Stahl and Bromme to collect data from university students in mathematics in the years of 2017, 2018 and 2019. The pseudo-longitudinal data analysis showed 1. that students hold different connotative beliefs regarding the two domains "mathematics at university" and "mathematics at school" regardless their study progress, 2. that the beliefs remain relatively stable within the domains overtime and 3. that – considering the different mathematical programmes of study (e.g., pre-service teachers vs. mathematics majors) – the students' connotative beliefs mainly differ regarding beliefs about the simplicity of mathematical knowledge at school.

Keywords: epistemological beliefs, connotative aspects, development, mathematics, pseudo-longitudinal study

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## 1 Introduction

Learners' beliefs about the nature of knowledge and knowing (Hofer & Pintrich, 1997), epistemological beliefs (EB), impact on the choice of learning strategies and information processing, i. e. the integration and acceptance of new knowledge, the comprehension of information, etcetera (Mason & Boscolo, 2004; Buehl & Alexander, 2002; Pintrich, 2002). Therefore, EB eventually affect students' learning outcomes (Schommer, 1993) what might be in terms of grades and rankings crucial for subsequent career options. Furthermore, epistemological beliefs presumably affect teachers' individual teaching styles, i. e. the choices of teaching methods and the subject-specific presentations of knowledge structures and knowledge justifications (Brownlee et al., 2011). The learners adopt the presented knowledge and way of knowing, and thus, teachers' EB indirectly shape their students' EB.

To assess and study EB in a differentiated and comprehensive manner one has to consider the impact of environmental factors on EB in two ways:

1. Environmental factors lead to domain-specific EB (e.g., Stahl & Bromme, 2007; Rott, 2020).
2. Environmental factors promote change and development of EB (e.g., Perry, 1970; Ross & Bruce, 2005).

This study addresses domain-specific EB (1) focusing on mathematics-specific beliefs and further differentiating EB regarding "mathematics as a school subject" and "mathematics as a scientific discipline" as taught in university. This differentiation is made due to the differences in contents, learning and teaching approaches as well as thinking and inquiry methods between subjects taught at school and taught at university (cf. Stengel, 1997; Bromme, 1994). Therefore, one might assume that university students adopt different EB concerning school subjects and related academic disciplines as a result of the leverage effect of the different teaching contents.

Regarding mathematics, "mathematics taught at school and at university [particularly] differ […] in terms of rigor and in the necessity that is seen for justification" (Dreher et al., 2018, p. 323; see also Beswick, 2012): Mathematics as a scientific discipline (scientific maths) primarily "focuses on the rigorous establishment of theory in terms of definitions, theorems, and proofs" (Dreher et al., 2018, p. 323). The prevalence of the axiomatic-deductive structure results from the mathematical community's request for warrants. Those are presented in journals, books, or lectures as proofs and deductively justified theorems although new concepts and ideas are usually not found in a deductive reasoning process (Ernest, 1999). In mathematics classrooms at school (school maths), new concepts are introduced rather empirically, for example by examining prototypes instead, and reasoning is rather context-related and intuitive than rigorous and abstract (cf. Dreher et al., 2018). Scientific maths usually operate on an abstract level using symbolic mathematical language whereas school maths puts emphasis on the practical benefits of mathematics in everyday life and mainly presents it as a tool to approach and analyse reality (ibid.).

Considering the gap between school math and scientific math, Beswick (2012) emphasizes the key role of mathematics teachers in reducing the gap, stating they can reduce the differences if "they have an appreciation of the nature of mathematics that is akin to that of mathematicians" (p. 129).

Based on this insight, Beswick collected and examined i. a. teachers' EB about the nature of mathematical knowledge. She asked eight mathematics teachers to respond to 26 items on a five-point Likert scale, conducted six semi-structured interviews, and observed teaching sessions. Although Beswick does not claim representativeness of her findings, the examined cases suggest that beliefs of experienced mathematics teachers may differ regarding school maths and scientific maths. New insights into the nature of scientific maths gained while studying at university, do not necessarily transform beliefs about school maths "rather adding up on beliefs from earlier schooling experiences" (p. 145).

Considering the varying educational effects of different study programmes, this study additionally examines the EB of students being enrolled in different mathematical study programmes (e.g., mathematics major studies or mathematics for upper secondary school teaching) regarding school maths and scientific maths.

Grigutsch et al. (1998) have already examined study-programme related differences of mathematical beliefs in the context of surveying "mathematical world views". They used a questionnaire completed by 310 teachers that asked to agree on different statements about the nature of mathematics on a four-point Likert scale. The statements described mathematics to be either a formal-coherent system, a process determined by activity, a schematic toolbox or formula set and expressed opinions on its range of applicability, amongst others. The researchers compared the data of different teacher degree courses. Most of the students agreed on the significance of rigor in mathematics and on a wide range of applicability of mathematics. They also considered mathematics to be a process-driven discipline. But most of the students denied mathematics being reduced to a schematic toolbox or a formula set except for students of lower secondary teaching who expressed a significantly higher rate of agreement on that schematic view of mathematics. Grigutsch et al. (1998) summarized that the mathematical world views, and thus, the mathematical beliefs, of students in different teacher-training programmes, do not differ substantially.

Besides building on the research done by Grigutsch et al. (1998) – considering not only EB of pre-service teachers but mathematical EB of science students as well –, this study tracks the development of EB in both domains, school maths and scientific maths, for three years of study at university, too, and thereby considers the second aspect of environmental factors (2). Research findings either characterise the development of EB in a normative way, describing a development from naïve towards sophisticated EB (e.g., Perry, 1970; Kuhn, 1991), or in a quantitative way, finding EB to be rather stable or unstable in the course of time (e.g., Charalambous & Philippou, 2003; Green, 1971). Liljedahl et al. (2012) attribute the inconclusive results about the stability of EB – being perceived as more stable or "more susceptible to change" – to the lacking common definition of belief stability. Several researchers even have assumed that both characteristics of beliefs are not necessarily "mututally exclusive" and e. g., suggest that EB consist of core and peripheral belief aspects (Green, 1971; Kaasila et al., 2005), with the latter ones being more changeable and the former ones being more stable.

The present studies neglect the normative evaluation of the development of EB (with a distinction between naïve and sophisticated EB) and focusses instead on the quantitative evaluation of EB development, i. e., assesses the stability of mathematical EB during studies at university. In respect of the inconsistent research findings on belief stability, this article examines a specific aspect of EB, the connotative aspect of EB, which is explained in the following. Thus, it continues the theoretical approach of distinguishing different belief aspects, this, tries to shed light on the inconsistencies of research of findings about belief stability, and provides a new perspective on the nature mathematical EB.

## 2 Theory

### 2.1 Denotative and connotative aspects of epistemological beliefs

Stahl and Bromme (2007) take the different components of EB (e.g., Green, 1971) and their influence in forming beliefs into account and differentiate between connotative and denotative beliefs. This terminology is inspired by linguistics in the sense that a connotative meaning of a word is an associated, usually culturally shared meaning in addition to its denotative meaning. The denotative content is the precise, propositional, literal sense of a word.

Accordingly, connotative aspects of epistemological beliefs (CEB) denote associative-evaluative assumptions about the nature of knowledge which tend to be spontaneous, more emotional, and personal (Stahl & Bromme, 2007). In terms of mathematics, for example, connotative judgments about mathematics are stimulated "when a student is asked whether he or she generally thinks that mathematical knowledge is rather certain or uncertain" (Rott et al., 2015, p. 40) and no further context is given. On the contrary, denotative aspects of epistemological beliefs (DEB) encompass explicit, reflected-upon knowledge about the nature of knowledge and often are less contextual (Stahl & Bromme, 2007) and can be grouped into naïve and sophisticated DEB (Rott, 2020). Despite of the suggested distinction, Stahl and Bromme (2007) do not assume CEB and DEB to be strictly separable from each other.

## 2.2 The CAEB (Connotative Aspects of epistemological beliefs) – an instrument by Stahl & Bromme

Based on Osgood and Snider's semantic differential (1969), Stahl and Bromme (2007) developed an instrument to measure the Connotative Aspects of Epistemological Beliefs (CAEB). Osgood et al. (1957) originally used the semantic differential as a quantification method for "affective meanings".

Stahl and Bromme (2007) collected 24 pairs of opposing adjectives to be judged on a 7-point Likert scale for each contrastive pair, describing EB in the dimensions: "(a) [...] simplicity of knowledge (knowledge consists of simple facts vs. it is a complex network of information), (b) [...] certainty of knowledge (knowledge is certain vs. it is tentative), and (c) [...] source of knowledge (knowledge is objective and observable vs. it is subjective and constructed)" (p. 775). For the purpose of validation, they tested their items in two studies with more than 1000 participants each and identified 17 stable adjective pairs via factor analysis which could be summed up under the two factors "Texture" describing beliefs about the structure and accuracy of knowledge and "Variability" describing beliefs about the stability and dynamics of knowledge. In those studies, the CAEB proved to be sensitive enough to detect differences in students' CEB about different domains.

## 2.3 CAEB adaptation and further findings by Rott, Leuders, & Stahl

Since several research results indicate that there are domain-dependent EB (De Corte, Op't Eynde, & Verschaffel, 2002; Hofer, 2000), Rott et al. (2015, 2017) aimed to measure CEB about mathematics on a discipline-specific level.

For this purpose, Rott et al. (2015) had 230 respondents complete the CAEB twice, once with "mathematics as a school subject" (school maths), the second time with "mathematics as a scientific discipline" (scientific maths) in mind. Rott et al. (2015) focussed on epistemological judgements about the certainty of mathematics and found ten items that could be subsumed under the factor "Certainty" via factor analysis. They also collected denotative judgments of students about the certainty of mathematical knowledge and a two-way ANOVA supported the distinction of connotative and denotative judgments. In comparison to DEB, the collected CAEB-data reveal nothing about the degree of reflection upon the claimed beliefs or the sophistication of the students' beliefs.

Rott et al. (2017) repeated the survey with 147 students (105 1st and 42 4th semester students). A factor analysis showed that three factors could be distinguished: Certainty/ Texture, Simplicity and Variability of mathematical EB. Using this three-factor-based model, Rott et al. (2017) compared CEB about school maths vs. CEB about scientific maths. They found that the students judged school maths to be significantly easier and more superficial than scientific maths. Moreover, the trend could be observed that scientific maths was perceived to be more tentative and variable. School maths, on the other hand, was judged to be more organized, but also to be more inaccurate. Looking at the study progress, more advanced students (in the 4th semester) rated school maths to be significantly more tentative and more structured and scientific maths to be significantly easier compared to the judgement of the first-year students.

## 3 Research objective

The focus of this article lies on the analysis of CEB as portrayed in the introductory section, and thereby continues the research done by Rott et al. (2017), trying to answer the following main question about the domain-dependent nature of CEB: How do CEB about school maths and scientific maths develop during three years of bachelor's degree at university? Or put more precisely: In what sense differ CEB about mathematical knowledge regarding school maths and scientific maths in different semesters of study?

Such differences in CEB regarding the mentioned domains have already been hypothesized by Rott et al. (2017) based on a small sample and are probable due to the different representation modes of mathematical knowledge in school and in university (see *Introduction*). As freshmen are not accustomed to scientific maths, their CEB about this domain might shift in the course of their studies. Furthermore, it may well happen that the students – especially those that are enrolled to become teachers – reassess their beliefs about school maths over time as they gain new experiences, new knowledge and new skills. Rott et al. (2017) have found that first semester students and 4th semester Bachelor students differ regarding their EB about school maths and scientific maths (see *Theory*).

To comprehensively analyse CEB, the mentioned research question includes and combines the two main environmental factors that determine the nature of mathematical CEB, namely 1) the domain-specific formation of EB and 2) the development of EB over time (see *Introduction*). Tracking the development of CEB is reasonable as a static snapshot might not reflect the overall nature of CEB. And surveying general, not domain-specific CEB might not represent the mathematics-specific nature of CEB.

In fact, we further investigate the differences in domain specific CEB and the beliefs' development with regard to the students' different programmes of study. This allows an even more differentiated insight into the impact of environmental factors on CEB as students of different programmes of study are trained considerably differently, i.e., they have to meet different educational requirements and specialise in different fields. For example, pre-service teachers for upper secondary schools attend the same mathematics courses as students of the study programme "Bachelor of science". Pre-service teachers for primary school and for lower secondary school", on the contrary, usually attend less demanding university courses in terms of mathematical skills and knowledge. Pre-service teachers for primary schools do not choose to study mathematics voluntarily; it is a mandatory part of their curriculum. Accordingly, the different educational requirements might account for the slight differences in beliefs found by Grigutsch et al. (1998) between students of different teaching programmes and might account for intraindividual differences in CEB as well (see chapter *Introduction*). Unlike students majoring in mathematics, all pre-service teachers learn about the didactics of mathematics in addition to university mathematics and remain connected to school maths as they go through practical training sessions at school during their studies. These curricular activities might affect the pre-teacher's beliefs about school maths and induce change of CEB in this domain over time.

Whereas many surveys focus on mathematics teachers' EB (while still in university training or working professionally), none to little surveys consider EB of students majoring mathematics compared to students in teacher training. In this respect, this study, e.g. extends the research of Rott et al. (2017).

## 4 Method

This study is part of the project "Learning the Science of Mathematics" (LeScMa), in which students' skills in mathematical critical thinking as well as DEB and CEB have been assessed (Rott, 2020); here, we focus on the latter. For the assessment of mathematical CEB during academic studies, the CAEB was presented to university students attending different mathematical programs of study at the University Cologne at the beginning of the winter terms in 2017, 2018, and 2019, respectively (cf. Schreck et al., 2023). 1774 students completed the questionnaire (601 male, 1086 female, 87 students did not specify their gender): 580 students in 2017 (mean age 20.36, SD 3.06), 397 in 2018 (mean age 22.03, SD 3.26), and 797 in 2019 (mean age 21.71, SD 2.95). 84 individuals participated in all three rounds of survey, 279 individuals participated twice either in 2017 and 2018, 2018 and 2019, or 2017 and 2019, and 1495 were single participants. Participants created pseudonyms which allowed to track single and multiple participation. 365 of the respondents were preparing to become upper secondary teachers, 127 students were lower secondary teachers, 412 students were primary teachers, 340 students were teachers for special needs, 428 of the respondents were students of the Bachelor of Science degree (mathematics majors). 150 did not specify their field of study.

The students completed the adapted CAEB questionnaire in 10-15 minutes during lecture time; participation was voluntary. The CAEB asks the students to position themselves on a 7-point Likert scale between two opposing adjectives describing two opposing epistemological beliefs. The semantic differential format combined with limited response time should ensure that the students judged the positions on an associative-connotative basis. 24 adjective pairs were given in total and the students responded to the CAEB twice, with regard to first school maths, and to second scientific maths. In a previous study by Groß Ophoff et al. (in prep.), the adjective pairs in the adapted CAEB-version could be classified into the factors “Texture/ Certainty”, “Variability”, and “Simplicity”.

**Knowledge in the domain of "mathematics as a school subject" is:**

| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | simple | ❑ | ❑ | ❑ | ❑ | ❑ | ❑ | ❑ | complex |
| 2 | stable | ❑ | ❑ | ❑ | ❑ | ❑ | ❑ | ❑ | unstable |
| 3 | dynamic | ❑ | ❑ | ❑ | ❑ | ❑ | ❑ | ❑ | static |

Figure 1. Excerpt of the adapted CAEB-questionnaire to survey CEB regarding school maths and alternatively regarding scientific maths.

The first factor, “Certainty/ Texture”, is a mixture of concepts about the nature of knowledge and about the nature of knowing operationalizing EB about the accuracy and safeguard of knowledge. The factor “Variability” represents beliefs about the stability and dynamics of knowledge (see Figure 1: the adjective pairs “stable vs. unstable” as well as “dynamic vs. static”). According to the analysis of CEB about the knowledge in Educational Sciences compared to Mathematics (as a common subject in teacher education) in a previous project, the originally proposed dimension Simplicity (Stahl & Bromme, 2007) could also be identified (Groß Ophoff et al., in prep.; see Figure 1: the adjective pair “simple vs. complex”). Therefore, the same psychometric structure, that had been validated via confirmatory factor analysis ($\chi^2 = 423.673$; df = 239; $\chi^2$/df = 1.8; CFI = .048; RMSEA = .918), was applied in this analysis: A multifactorial variance analysis was conducted for the self-reported beliefs about the “Certainty/Texture”, “Simplicity”, and “Variability” in the two separate domains school maths and scientific maths as dependent variables. The ratings about school maths and scientific maths were treated as repeated measurements as they were surveyed with parallel questionnaires at the same time of measurement. Furthermore, the three study programmes (Bachelor of Science, Mathematics for teaching at the upper secondary school, Mathematics for teaching at other German school forms) were used as independent variables. Academic progress was included as a covariate.

## 5 Results

Looking at the total sample (i.e., students of different study programmes and different semesters), school maths and scientific maths are perceived as two separate domains of knowledge (see Table 1; cf. Schreck et al., 2023). The largest discrepancy is found with the assessment of the factor “Simplicity” of knowledge. Correspondingly, school maths is judged to be significantly simpler (mean value (MV) 3.41 vs. 6.4) and more superficial (MV 3.52 vs. 6.23) than scientific maths. Judgments about school maths and scientific maths also slightly differ regarding the certainty, acceptance, precision, and confirmability of knowledge. Scientific maths is judged less certain (MV 3.26 vs. 2.8), less stable (MV 3.6 vs. 2.96), and more disputed (MV 3.48 vs. 2.83) than school maths, whereas scientific math is more precise (MV 2.41 vs. 3.07) and better confirmable (MV 2.28 vs. 2.65).

**Table 1.** Mean values of the item-ratings in the semantic differential of the CAEB-questionnaire. A wide cross section of all students that were surveyed in 2017, 2018, 2019. The adjective pairs are clustered into the three factors "Certainty/ Texture", "Simplicity" and "Variability".

| **Factor** | **Likert-scale rating options** | | **School maths** | **Scientific maths** |
|---|---|---|---|---|
| | **1 vs. 7** | | **Mean (Standard Error)** | |
| **1. Texture/ Certainty** | stable | unstable | 2.96 (0.034) | 3.6 (0.045) |
| | confirmable | unconfirmable | 2.65 (0.039) | 2.28 (0.036) |
| | exact | vague | 3.1 (0.035) | 2.86 (0.044) |
| | absolute | relative | 3.55 (0.034) | 3.4 (0.04) |
| | precise | imprecise | 3.07 (0.035) | 2.41 (0.033) |
| | definite | ambiguous | 2.6 (0.031) | 2.92 (0.038) |
| | accepted | disputed | 2.83 (0.037) | 3.48 (0.04) |
| | certain | uncertain | 2.8 (0.032) | 3.26 (0.036) |
| **2. Simplicity** | simple | complex | 3.41 (0037) | 6.4 (0.026) |
| | superficial | profound | 3.52 (0.039) | 6.23 (0.03) |
| **3. Variability** | dynamic | static | 4.56 (0036) | 3.82 (0.044) |
| | flexible | inflexible | 4.51 (0.36) | 4.29 (0.041) |

With regard to their study programmes, the students were sorted into three groups (students of the Bachelor of Science degree, pre-service teachers for upper secondary school, other pre-service teachers including pre-service teachers for lower secondary schools and for primary schools) to analyse study group specific CEB as well (cf. Schreck et al., 2023). Regarding school maths (see Figure 2), students aiming at the Bachelor of Science degree find school maths less confirmable (MV 3.04) and slightly vaguer (MV 3.38) while at same time more static (MV 4.75) than pre-service teachers of mathematics (see Figure 2). Pre-service teachers for upper secondary school judge school maths to be quite certain with the lowest rating of all students (MV 2.58), as well as well accepted (MV 2.63). They take fairly similar views on the stability (MV 2.68 vs. 2.8), flexibility (MV 4.4 vs. 4.32) and precision of mathematical knowledge (MV 3.26 vs. 3.39) at school as students enrolled in the "Bachelor of Science" study programme, whereas sharing similar judgements about the exactness (MV 3.02 vs.

3.02), confirmability (MV 2.72 vs. 2.47) and the dynamic nature (MV 4.47 vs. 4.5) of mathematics at school with students of other teacher training programmes. On the contrary, these latter pre-service teachers judge school maths most unstable (MV 3.13), confirmable (MV 2.47), disputed (MV 3.0) and uncertain (MV 2.93) but at the same time most inflexible (MV 4.68) of all study groups. All student groups rate the absoluteness (MV 3.61 vs. 3.46 vs. 3.56) and definiteness (MV 2.68 vs. 2.49 vs. 2.59) of school maths nearly the same while differing on the factor “Simplicity of knowledge”:

Students of the Bachelor of Science degree rate school maths to be the easiest (MV 2.54) and most superficial (MV 2.73), followed by the pre-service teachers for upper secondary school who rate it second easiest (MV 2.97) and second most superficial (3.28). Other preservice teachers cannot really decide on judging it rather easy or rather complex (MV 4.01), rather superficial or rather profound (MV 3.98).

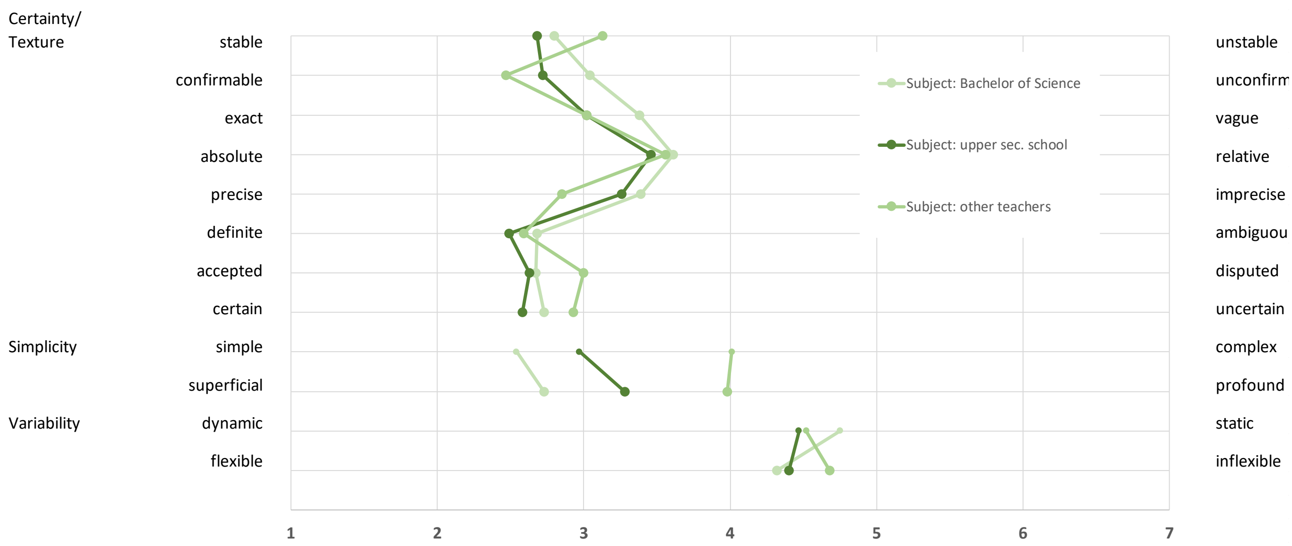


Figure 2. Item-rating “mathematics as a school subject” in the semantic differential of the CAEB-questionnaire. Ratings grouped by study programmes.

The judgements about scientific maths tend in the same direction regardless the students’ study programme and the considered item-factor (see Figure 3; cf. Schreck et al., 2023). And yet, students of the study programme “Bachelor of Science” rate items regarding the factors “Certainty/ Texture” and “Variability” the lowest, which means that they judge scientific maths the most stable (MV 3.34), precise (MV 2.27), confirmable (MV 2.12), accepted (MV 3.18), certain (MV 3), dynamic nature (MV 3.56), and the most flexible (MV 4.05). The students of all study groups find scientific maths quite complex (e.g., Bachelor of Science MV 6.38) and profound (e.g., Bachelor of Science MV 6.24). Pre-service teachers apart from the pre-service teachers for upper secondary school are somewhat doubtful of the validity of mathematical knowledge at university (item “accepted vs. disputed” MV 3.68).

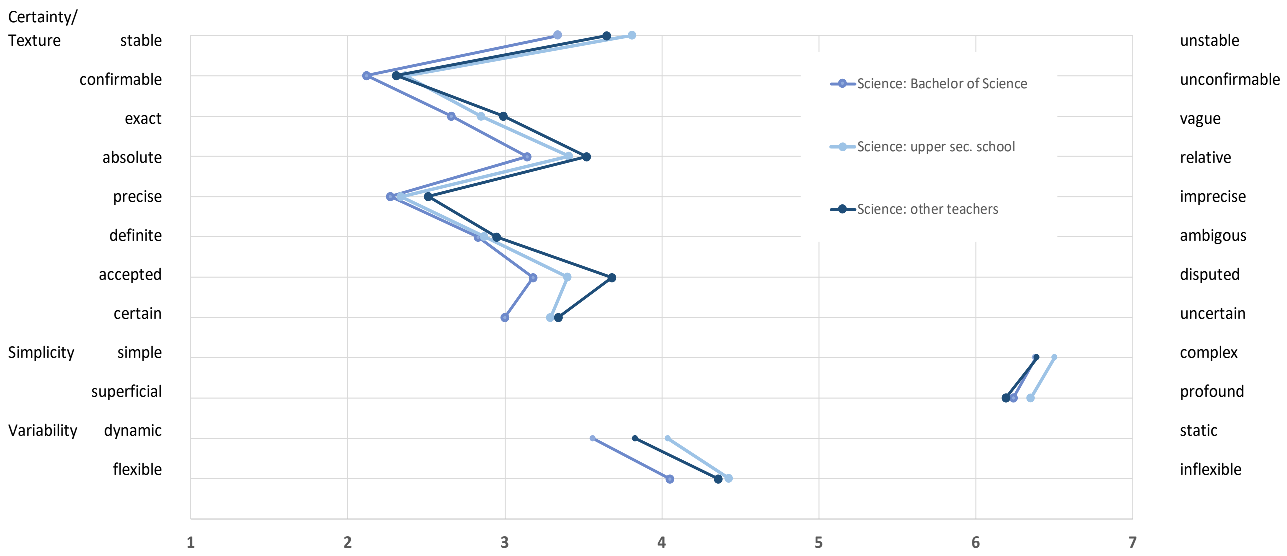


Figure 3. Item-rating “mathematics as a scientific discipline” in the semantic differential of the CAEB-questionnaire. Ratings grouped by study programmes.

To trace possible developments during bachelor’s degree, the participants—regardless of their programmes of study—were sorted into three groups: 1st and 2nd semester, 3rd and 4th semester, as well as 5th and 6th semester (students with a higher semester count were discarded for this analysis) (N=1493; cf. Schreck et al., 2023). The data do not suggest great change in connotative judgements about school maths within the first three years of university studies (see Figure 4). The greatest shifts occur regarding the factor “Simplicity” whereby school maths is rated to be slightly simpler (MV 3.2 vs. 3.37) and more superficial (MV 3.28 vs. 3.74) by 5th and 6th semester students compared to 1st and 2nd semester students.

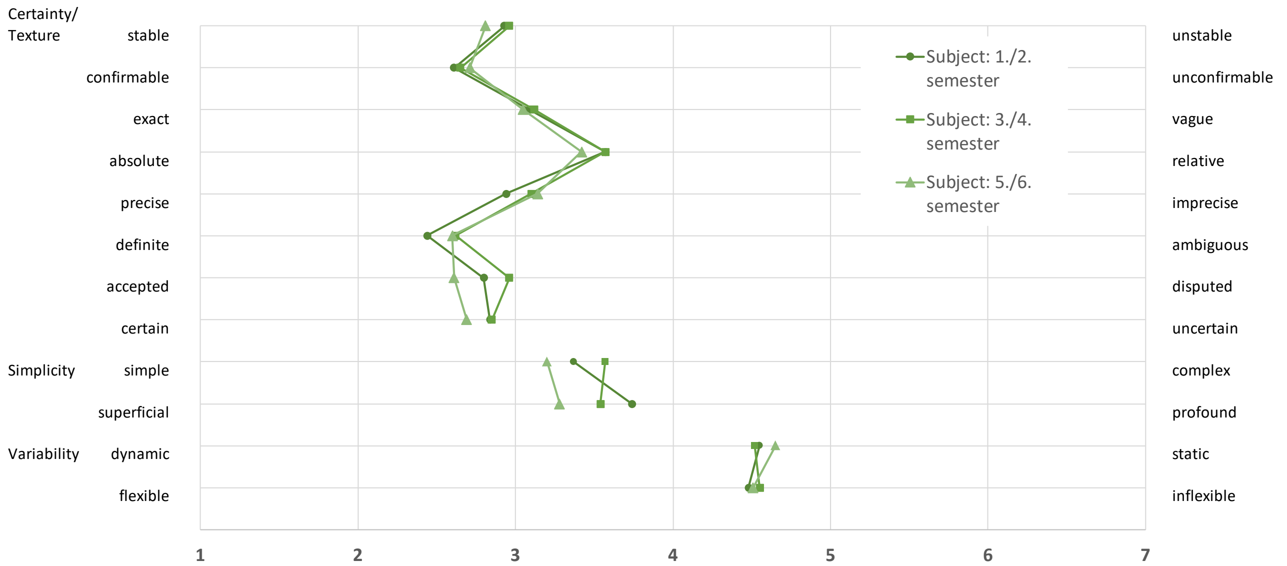


Figure 4. Item-ratings on school maths in the semantic differential of the CAEB-questionnaire. Ratings grouped by semester of study.

Nearly the same is true for the ratings on scientific maths, given by students of the three groups in different semesters (see Figure 5; cf. Schreck et al., 2023): The connotative judgements about mathematics at university shift surprisingly little during studies. The greatest change can be seen regarding judgements about the certainty and acceptance of mathematical knowledge. Thus, 5th and 6th semester perceive "mathematics at university" as more accepted (MV 3.35 vs. 3.54) and more certain (MV 3.09 vs. 3.39) than 1st and 2nd semester students.

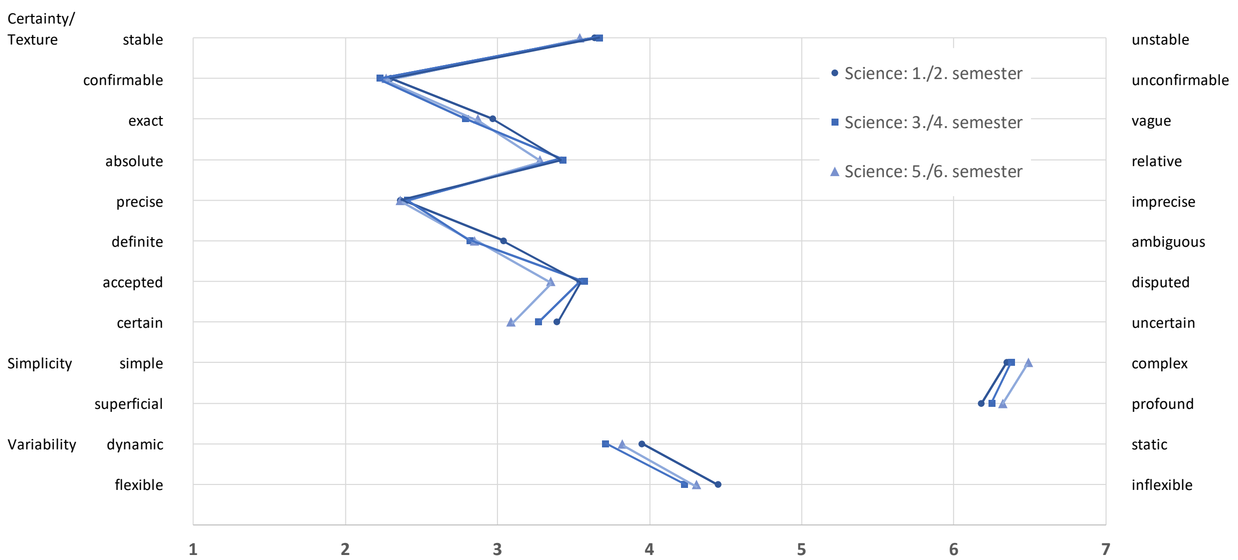


Figure 5. Item-ratings on scientific maths in the semantic differential of the CAEB-questionnaire. Ratings grouped by semester of study.

**Table 2.** The multi-dimensional variance analysis with repeated measures found statistically significant main effects for the self-reported beliefs ($F(1,1566) = 513.738$, $p \leq .05$; $\eta^2 = .396$; Wilk's $\Lambda = .604$), for the two domains school maths and scientific maths ($F(1,1566) = 571.572$, $p \leq .05$; $\eta^2 = .267$; Wilk's $\Lambda = .733$), and for the between-subjects factor "study programme" ($F(1,1566) = 176.853$, $p \leq .05$; $\eta^2 = .087$) (see Table 2).Item factors vs. programme of study vs. domain: The largest discrepancies concerning the two domains school maths and scientific maths can be found with the factor "Simplicity" regardless the programme of study. The judgements regarding the other two factors do not differ much.

| Belief factor | study programme | mathematics as school subject MV and SD | mathematics as science MV and SD |
|---|---|---|---|
| Certainty/ Texture | Bachelor of Science | 3,0 (0.9) | 2,8 (1.2) |
| | Upper sec. school | 2,9 (0.9) | 3,0 (1.2) |
| | Other schools | 2,9 (0.9) | 3,1 (1.1) |
| Simplicity | Bachelor of Science | 2,6 (1.2) | 6,3 (1.0) |
| | Upper sec. school | 3,1 (1.3) | 6,4 (1.0) |
| | Other schools | 4,0 (1.2) | 6,3 (1.0) |
| Variability | Bachelor of Science | 4,5 (1.3) | 3,8 (1.4) |
| | Upper sec. school | 4,4 (1.3) | 4,2 (1.6) |
| | other schools | 4,6 (1.2) | 4,1 (1.4) |

Furthermore, significant, but rather small interaction effects emerged for

- domain vs. programme of study ($F(2,1565) = 41.772$, $p \leq .05$; $\eta^2 = .051$; Wilk's $\Lambda = .949$)
- connotative judgements vs. programme of study ($F(4,3130) = 23.824$, $p \leq .05$; $\eta^2 = .030$; Wilk's $\Lambda = .942$)
- domain vs. connotative judgements ($F(2,1565) = 397.252$, $p \leq .05$; $\eta^2 = .337$; Wilk's $\Lambda = .663$)
- domain vs. connotative judgements vs. academic progress ($F(2,1565) = 5.4$, $p \leq .05$; $\eta^2 = .007$; Wilk's $\Lambda = .993$)
- domain vs. connotative judgements vs. programme of study ($F(4,3130) = 41.981$, $p \leq .05$; $\eta^2 = .051$; Wilk's $\Lambda = .901$)

No significant effects were identified for the covariate "academic progress" ($F(1,1566) = .001$, $p > .05$) or the interaction effects of CEB vs. academic progress ($F(2,1565) = .169$, $p > .05$; Wilk's $\Lambda = 1$) or domain vs. academic progress ($F(1,1566) = 3.392$, $p > .05$; Wilk's $\Lambda = .998$).

## 6 Discussion

We conclude about the nature of CEB that all students regardless of their study programme or academic progress hold different beliefs about school maths and scientific maths (cf. Schreck et al., 2023). Therefore, we assume that students perceive these domains as separate domains of knowledge. Especially, differing beliefs about the simplicity of knowledge in both domains (see Table 1 & 2) indicate the discrepancy. The causes of such a domain-sensitivity of specific CEB need to be discussed and further investigated: One plausible explanation for the domain-sensitivity of the belief factor "Simplicity of knowledge" in the present study is that the given research design esp. responding to the questionnaire twice successively – once with school maths and once with scientific in mind –, might have enhanced a contrast effect in respect of the ratings on the simplicity of mathematical knowledge in the two domains. One practical way to mitigate such a probable contrast effect of the two questionnaires might be to ask the students to respond to the questionnaire twice separately – regarding school maths and scientific maths – with larger time lags in-between. At least, students *of different study programmes* particularly judge the simplicity of mathematical knowledge at school differently (see Figure 2) which could reflect a selection effect taking place with the choice of the study programme at beginning of studies. That means that students who are enrolled in the mathematically more demanding study programmes assumably found mathematics at school to be comparably simpler than their classmates having a natural affinity for mathematics. Another reason might be that pre-service teachers are more preoccupied with learners' difficulties with school maths as they learn about those difficulties in practical training sessions at school and in the field of didactics during their studies.

Looking at the development of beliefs, CEB about school maths and scientific maths prove to remain relatively stable within the domains respectively throughout the course of the participants' bachelor's programme (see Figures 4 & 45) which means particularly that the anticipated shift in beliefs about scientific maths and school maths did not occur over time. Therefore, we assume that the CEB of the students were quite resilient to the students' current social, emotional context or surrounding environment at the given times of measurement. Besides, considering environmental stimuli for belief change, the little shift in judgements, e.g., about the belief factor "Variability", might result from little to none discourse about mathematics at the boundaries of knowledge during school education as well in the first academic years. Accordingly, Ross and Bruce (2005) claim that there must be great environmental stimulus to induce change in beliefs over time, at least in terms of pre-service teachers.

Appropriately, besides the shown temporal stability of CEB, it would be interesting and useful to investigate the stability of *denotative* aspects of epistemological beliefs as well, to examine whether and in which manner structural aspects of beliefs contribute to the claimed simultaneous maintenance of stable and flexible beliefs, for example.

Unlike school maths, scientific maths is rated to be highly complex and profound by students of all semesters and study programmes (see Figures 3 & 35 ). These divergent CEB regarding scientific maths might result from the discrepancies between mathematics teaching in school and at university (see Introduction). Thus, difficulties with the subject matter at university and the corresponding CEB about the simplicity of scientific maths may arise from the fast pace of progression in lectures and seminars, the huge amount of study matter, greater complexity of the subject matter, the high level of abstraction of advanced mathematics, the continuous demand for rigor and proof in lectures and seminars, the students' own responsibility for their learning progress, necessary skills regarding self-organisation and time management, etc.

A limitation of this study is that due to the CEB, a new aspect of EB in educational research, comparisons to previous studies on beliefs from the respective literature might fall short.

Finally, even though data was gathered in three consecutive years, the study at hand is not a longitudinal study in the narrow sense, i. e. tracing the EB of individual students from the 1st to the 3rd to the 5th semester of their bachelor's degree. Instead, we use a pseudo-longitudinal or panel approach, to have a large enough number of participants to interpret the quantitative data. The analysis of the actual longitudinal data (cf. Schreck et al., 2023) confirmed the results described above: 1) different judgement of school maths and scientific maths in general, 2) different judgements by different study groups especially regarding the simplicity of school maths, 3) domain-wise stability of the judgements over time.

## Note

This article has a slight overlap with the article "Studying mathematics at university level: a sequential cohort study for investigating connotative aspects of epistemological beliefs" published in the International Journal of Mathematical Education in

Science and Technology (2023), as both report on the same project..

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# How to deal with and utilize (mathematics (education)) researchers’ beliefs

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This paper addresses the desideratum identified by Törner (2018), that researchers' beliefs are rarely addressed in the research literature dealing with beliefs. For this purpose, firstly a suitable theoretical framework is outlined that links the concept of belief with the research perspectives of researchers. Secondly, examples are given of how beliefs were, can and should be addressed in corresponding research on beliefs. Finally, it is shown in which ways explicating beliefs of mathematics education researchers might made their research, as well as their teaching more effective.

Keywords: researcher beliefs, background theory, methodology, domains of subjective experience

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## 1 Introduction

Deciding on the topic for my contribution at the MAVI-conference this year, I thought which topic might be interesting for the research community. My dissertation project (Stoffels, 2020) was based on the ÜberPro_WR seminars, which were designed to foster the reflection of students’ own beliefs on mathematics during their transition from school to university and comparing them with the beliefs which were held during the transition to formal probability theory in its historical development in the 20th century. During the seminars, I became more and more aware of how important it is to make one's own beliefs on mathematics and probability theory as teacher and researcher explicit in these seminars to promote the students' reflection on their own beliefs. However, making my beliefs explicit does not mean that students were forced to simply adopt these beliefs, but rather to create an awareness of multiple perspectives on mathematics and probability theory. As a result of this observation, I have planned to choose the topic "why you can learn a lot about researchers' beliefs analysing their research on mathematical beliefs" for this contribution. Unfortunately, or rather fortunately, there is already an article by Törner (2018) that deals with similar issues that I had in mind, in particular:

- Describing the state of research: “It should have been pointed out that in research literature dealing with beliefs, researchers’ beliefs are often neglected.

This may be due to the assumption that researchers should not be accused of having beliefs in the first place. Beliefs are regarded as features of subordinate teachers, students, parents, educational administrators, and further stakeholders, but not as features of researchers." (Törner, 2018, p. 7)

- Reflecting the state of research and recommendation for further research: "In research literature, this lack of self-reflection is hardly ever mentioned. We believe that this can be regarded as a 'blank spot'." (Törner, 2018, p. 7)
- Indicating (mathematics education) researchers' beliefs influencing educational practice: "This circumstance is tragic since researchers have to be seen as important players in terms of educational change." (Törner, 2018, p. 7)

So instead of raising these issues I decided to tackle them by providing a 'how-to' guide based on theoretical considerations as well as empirical indications how to deal with and utilize ((mathematics (education)) researchers') beliefs[1].

## 2 How to frame (researchers') beliefs: interaction vs. reflection

There are a lot of different descriptions and definitions of the belief concept in literature (Goldin, 2003; Green, 1971; Grigutsch, Raatz & Törner, 1998; Pajares, 1992; Schoenfeld, 1985; Stoffels, 2020; Thompson, 1992), furthermore a lot of research works mention that there is no consent on a definition of belief (Bräunling, 2017; Pehkonen, 1995; Rolka, 2006). Still, there seems to be no satisfactory answer to these theoretical problems; instead, a lot of research dealing with beliefs focus on exploring beliefs of different bearer groups, e.g., teachers and students (Törner, 2018), or the diversity of beliefs in different mathematical fields, establishing the concept of domain-specific beliefs (Eichler & Erens, 2015; Witzke & Spies, 2016).

An interesting discussion of the "theoretical struggles" is given by Goldin et al. (2009) referring to different perspectives and uses of the term beliefs, stating:

[1] In the title, throughout the article, and even in the term "((mathematics (education)) researcher's) beliefs" marked by this footnote, there is a bracketing notation that at first seems odd. However, it is meant to indicate two things. On the one hand, different belief-bearers, namely, unidentified belief-bearers, researchers in general, mathematicians, and mathematics education researchers, are considered by omitting the bracketed words. On the other hand, this notation is used to illustrate that while beliefs may differ content wise, they do not differ in terms of their structure and development as presented here.

> "Beliefs are highly subjective and vary according to the different bearers. Thus, observers of a specific situation may refer to quite different beliefs. [...] Our goal is to be able to apply the flexible construct of beliefs to various situations pertaining to mathematics education". (Goldin et al., 2009, p. 4)

Furthermore, they describe four classes of aspects of beliefs (Goldin, Rösken & Törner, 2009, p. 4) which are the ontological aspects (referring to a "belief object"), enumerative aspects (a "content set" of mental states or experiences connected to beliefs), normative aspects (how conscious the belief bearer is about the activated belief) and affective aspects of beliefs ("emotional feelings, attitudes and values" attached to the belief).

It seems that these attributes and aspects of beliefs are commonly accepted, especially as they are relatively general in nature. In relation to Törner's (2018) original desideratum regarding the beliefs of mathematics education researchers and this paper, one can conclude, assuming the premise, that the beliefs of researchers are not fundamentally different in nature from those of other bearers of beliefs (e.g., students or teachers), the following framing of researcher's beliefs might work. It simply adds the word (mathematics (education)) researcher to Goldin's et al. (2009, p. 4) description:

1. (Mathematics (education)) researchers' beliefs are highly subjective.
2. Beliefs vary according to the different (mathematics (education)) researchers.
3. (Mathematics (education)) researchers observing specific situations may refer to quite different beliefs.

Especially the third attribute shows the difficulties in the research on beliefs, as "observing specific situations" means, that during/after observations (mathematics (education)) researchers identify, probably a better word may be 'assign', beliefs to the observed bearer of beliefs while referring to their own beliefs. So, it seems as if there are multiple levels of beliefs, beliefs about beliefs, and so on.

Before these statements are elaborated further and explained by relating beliefs and belief-systems to the theory of "domains of subjective experience", some concrete examples of these statements will be given here.

For the highly subjectiveness of (mathematics (education)) researchers' beliefs, I will give two examples, an explicit and an implicit one. The explicit example is given by Grigutsch et al. (1998, pp. 13-14)) who stated that their qualitatively identified four aspects of "mathematical worldview" ('schema', 'formalism', 'process' and 'application') may have their origin in their own worldview. Implicitly the subjectiveness of beliefs can be stated by the various catalogues of aspects or belief-systems, which can be found in literature, based on previous experiences and backgrounds of the authors (Ernest, 1989; Grigutsch et al. 1998; Beswick, 2012). For the second and third claim I want to give a paper (Heyd-Metzuyanim, 2019) as an illustrative example, because it discusses different background theories, which are based on different beliefs about whether internal or "mental" constructs are fruitful for addressing different research questions in mathematics education. Or, as Heyd-Metzuyanim (2019, p. 7) states:

> "as exemplified in the two studies reviewed above, studies of beliefs and identity tend to crossover and deal with aspects that belong, according to the above suggestion, in the *others' camp* [emphasized by G.S.]".

Goldin et al. (2009, p. 3) referring by their construction of "constitutive elements of a structural framework guiding our understanding of beliefs" on Hilbert's (1899) approach to axiomatization in his "Foundations of Geometry" as implicit defining mathematical concepts in comparison to the classical definitions of "points" and "lines" by Euclid. The similarity, according to the authors, is that the constitutive elements of beliefs they propose provide a framework for discussing different perspectives on beliefs, just as Hilbert's implicit definitions in his axiomatics can do for explicit definitions in the context of geometry. In the following I will proceed analogously to particle physics and show in which way established concepts as "society of the mind" (Minsky, 1988) used by mathematics education and mathematics educational theories like "domains of subjective experience (DSE)" (Bauersfeld, 1983) can be a basis for the conception of belief in order to address the problem of researchers' beliefs of beliefs.

Similar theoretical issues to Goldin et al. (2009) are stated in Stoffels (2020) in the context of "Auffassungswechsel", which can be translated as "change of belief systems", in the transition from school to university. Considering similar to Goldin et al. (2009) (a) beliefs and belief-systems are subjective, which means they are internal, and (b) therefore it might be the case that observed participants may refer to different beliefs even in similar situations, following question arose: in which ways can a researcher indicate whether the observed participant refers to one, multiple, different, or similar beliefs? The idea solving this problem, that has guided my work, is that researchers *do not identify beliefs in bearers*, but rather *they attribute certain beliefs to them as observers of their activities in their environment*. This may look at first glance like a mere shift of the problem, but in the following it will become clear how this can be used productively for research on beliefs.

Still, researchers want to talk about assigning beliefs to observed subjects based on their activities, which may be guided by cognitive, affective, or behavioural processes (Liljedahl & Oesterle, 2014). Thus, according to such an interpretation of the belief concept, a theory is needed that explains the activities of the subjects in such a way, that it:

- allows the identification of different beliefs since mathematics educational research has made great progress in this area,
- can depict the above stated aspects of beliefs (Goldin et al., 2009), and finally.
- can form a basis for (inter-)active processes such as reflecting and sharing beliefs or doing/having a change of beliefs.

A good candidate for such a theory is offered by Bauersfeld's (1983) approach of subjective domains of experience (DSE). This insight is not fundamentally new since Pehkonen already described that in:

> “Germany, researchers usually speak instead of beliefs (Vorstellungen) and conceptions (Auffassungen) on "subjective theories" (e.g., Bauersfeld, 1983; Jungwirth, 1994; Tietze, 1990), and the central term to be used there is "a subjective experience domain" (Bauersfeld, 1983).” (Pehkonen, 1995, pp. 10–11)

A similar overview can be found in Grigutsch et al. (1998).

A new perspective can be established by using the DSE model as a suitable basis for the concept of belief and not as a mere similar concept (Stoffels, 2020). Accordingly, at this point I would like to first give a short overview on Bauersfeld’s (1983) conception of DSE, before I show that Goldin et al.'s (2009) aspects of beliefs can be found in the conception of DSE. Then I will give a definition of beliefs based on the DSE model, which allows explaining the reflection of beliefs as a relevant process for addressing beliefs. Finally, the issue of researchers’ beliefs on beliefs will be discussed.

The research in the 1980s and 1990s by Bauersfeld and his research group can be subsumed under the paradigm of Interactionism, which was also influenced by a long term cooperation with Paul Cobb (Cobb & Bauersfeld, 1995). In his working group several theoretical approaches were discussed how to shape this interactionist perspective. In this article I want to focus on two complementary foundations for the interactionist perspective Bauersfeld mentioned himself in his 1983 article “Domains of

Subjective Experiences as the Basic Issue for an Interactive Theory of Mathematics Learning and Teaching”. Bauersfeld states (1983, p. 40, translated by G.S.):

> “The DSE model allows for the clarifications of the concepts of abstraction, transfer, and illustration [...]. In particular, it allows a differentiated description of mathematical learning via the formation of new DSE and the linking of existing DSE. The frame model leads to a more precise description of institutionalized communication processes, in particular through terms like ‘working interim’, ‘frame conflict’, the phenomenon of ‘down-modulating’, etc.”

This juxtaposition shows Bauersfeld's assessment of the DSE model as an individualistic model. Stoffels (2020) has shown through a theoretical analysis based on an enactivist paradigm, that by considering the shared domain of experience of interactants, this individualistic limitation can be resolved, which is also important for this article. Apart from this extension of the DSE model in interactions, this article follows Bauersfeld's general conception of DSE, which includes the following main ideas (translated by G.S.):

- every subjective experience is domain-specific, i.e., a subject's experience is divided into DSE that are activated in the respective situations. (Bauersfeld, 1985, p. 11)
- the totality of DSE presents itself in an agglomeration of non-hierarchically ordered DSE - the "society of mind" (Minsky, 1988). The DSE compete for activation, the more effectively, the more frequently they are reactivated or the more intensively they have been formed. (Bauersfeld, 1985, p. 12)
- the crucial basis for the formation of a DSE is the subject's actions and the context of meaning he or she constructs, or more precisely, their formation in social interaction. (Bauersfeld, 1985, p. 14)
- since experience is total, a DSE includes various elements. Bauersfeld (1983, 1985) proposed a list of specific elements capable of being extended: knowledge, mathematical habitus, procedural knowledge, emotions, values, I-identity, etc.

In Table 1 the description of DSE by Bauersfeld (1983) is deconstructed for depicting corresponding specific elements of DSE for each aspect of belief by Goldin et al. (2009). Probably the most important and most frequently referred property of DSE is its domain specificity, which Bauersfeld (1983, p. 28, translated by G.S.) describes as “the ‘domain’ is less universal than a world. Just the limitedness and particularity separate the domains of subjective experience (by short as DSE) from each other”, which gives the possibility of assigning different and even contradicting beliefs to one person.

**Table 1.** Deconstruction of the DSE concept (Bauersfeld 1983, p. 17, 28, 56) for a comparison to the aspects of beliefs proposed by Goldin et al. (2009, p. 3)

| Aspects of belief | Domains of Subjective Experience |
|---|---|
| Ontological aspects: *belief object* | Perspectives and functions of DSE [Bauersfeld refers with these concepts on Lawler's (1981) microworlds, G.S.] |
| Enumerative aspects: (subjective) *content set* of various possible perceptions, characteristics, suppositions, philosophies, and/or ideologies, which are often simply referred to as beliefs, or better, belief states. | The mathematical habitus is a specific element of DSE. |
| Normative aspects: Beliefs are highly individualized, means that the elements of the content set possess different weights that are attributed to various perceptions or assumptions. | The designation [of a DSE by a researcher, G.S.] contains the reference to the 'subject' as bearer. |
| Affective aspects: beliefs are interwoven with affect – emotional feelings, attitudes, and values | [The concept DSE] focus 'total experience' and not only knowledge. The non-cognitive dimensions of motor skills, procedural knowledge, emotions, evaluations, identity, etc. are specific elements of DSE. |

Considering the distinction between DSE as situated in the subject and beliefs as assigned to the subjects by an observer (this could be a researcher) the following definition of beliefs can be given:

> A belief system (cf. Figure 1, dotted lines) refers to different domains of subjective experience (cf. Figure 1, filled shapes) that contain the same or similar perspectives and functions for the subject (cf. Figure 1, black ellipse and rounded rectangle). The clustering of reconstructed domains of subjective experience into belief systems on the basis of an identified sameness or similarity is done by an observer of the subject. This identification can be described as belief systems are clusters of domains of subjective experience. One way to specify this observation is to state that belief-systems of a subject form equivalence classes of domains of subjective experience of the subject. (Stoffels, 2020, p. 153, translated by G.S.)

The identified beliefs are therefore observer-related, which refers to the paradigm of enactivism (Maturana & Varela, 2008; Steinbring, 2015) in this conception. This does not mean that assigning beliefs is purely subjective by the observers. For example, there may be a mode of assigning beliefs by researchers according to methodological and content criteria that the scientific community considers as adequate. Examples might be the use of certain Likert-scalable items and an associated factor analysis, or a qualitative content analysis using theoretically grounded categories. This is for example the belief of researchers, that the four aspects provided by Grigutsch et al. (1998) are a reasonable choice or that the methodology given in this article is adequate. This belief can be reconstructed by showing the adaption of the aspects and methodology by other researchers (Schukajlow, Rakozy & Pekrun, 2017; Rolka, 2006).

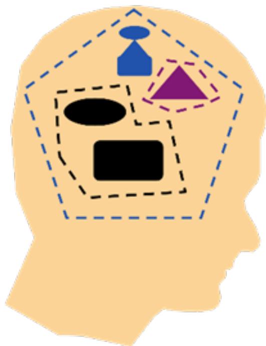

Figure 1. Diagram of different DSE's (black, purple, and blue filled figures) within the "society of the mind" (entire yellow polygon) and different indicated beliefs (dashed polygons). (cf. Stoffels, 2020, p. 154)

So, for the identification of beliefs an *interaction* between observer and subject is necessary. Of course, only those beliefs can be identified by the observer that he is aware of himself. Thus, one can speak of a *reflection* of one's own beliefs, in which the observer focus at his own activated perspectives and their functions. He thus interacts in a certain way with himself, which productively turns the previously identified theoretical issues of the researcher's beliefs about beliefs into a prerequisite for identifying beliefs. The required awareness of beliefs is implicitly shown by the fact that the observing researcher can take (different) perspectives into account, name them and thus indicate them. This can be also illustrated by Grigutsch et al. (1998), as they reflected their own beliefs on mathematics for getting an idea of possible beliefs on mathematics, they might be able to reconstruct for their participants in the study. In terms of DSE, this is only possible if the observer has at least one superordinate DSE whose perspectives allow the observation of subordinate DSE (Bauersfeld 1985, p. 40). In Figure 1, this is illustrated with the blue-filled figure, which diagrammatically represents a superordinate DSE that enables perspectives on the black- and purple-filled subordinate DSE. For example, considering Grigutsch et al. (1998) again, if they would have not a DSE for comparing the different aspects of beliefs on mathematics, they would not be able to compare this different aspects, but might be only able to activate the different beliefs in different situations.

An answer to the question of this paragraph's heading 'how to frame (researchers') beliefs' is then, that there is no need for a new or different frame, rather it seems to be more important making the beliefs explicit in the research work by reflecting own researcher's perspective as enablers of research dealing with beliefs (cf. Heyd-Metzuyanim, 2019).

## 3 How to fill the "blank spot" of research on researchers' beliefs: reconstruction vs. construction

Törner (2018, p. 7) comes to the conclusion, that there is a lack of self-reflection leaving out the self-reflection of the researchers, which can be "regarded as a 'blank spot'". Rather, I think there is a 'blind spot' in the literature which looks like as if there is a blank spot. So, in this paragraph I want to give some examples of how researchers' beliefs were, can and should be addressed in corresponding research on beliefs. So that researchers' beliefs becoming the object of belief.

One of the more classic examples I already used in the previous paragraphs can be found in Grigutsch's et al. (1998, pp. 13–14) three reasons why they considered the well-known four aspects ('schema', 'formalism', 'process' and 'application') of mathematical world views. In this paper I want to focus on the first part of the first reason as it neither refer to their theoretical analysis (reason 2) nor their empirical results (reason 3):

> "We believe that these four aspects are the central and strategic elements of mathematical worldviews. Possibly this is an expression of our own worldview. But multiple observations – which are certainly also selectively guided by our own attitudes – showed us that thinking about mathematics and mathematical teaching often occurs in these four dimensions." (Grigutsch et al., 1998, p. 13, translated by G.S.)

Using the perspectives of the previous paragraphs we can *reconstruct* multiple beliefs by indicating that Grigutsch et al. (1998) activated a superordinate DSE, which allowed them to address four different perspectives on mathematics, respectively doing mathematics, which may be originated from subordinate DSE only allowing separate perspectives on each aspect.

One has to speak of ‘may’ here, because due to the same conditionality, which Grigutsch et al. (1998) address in this quote, one's own evaluation is also determined by one's own experiences and perceptions. Also, it is part of Bauersfeld's (1983) concept of DSE, that DSE can only be reconstructed interpretively and incompletely, because of its situatedness in the subject. This is true, even if the observer is the subject him-/herself, insofar as the DSE cannot be recognized ‘completely’, since this would require another superordinate DSE, which the subject needs to activate as a reflecting observer.

Presumably, all research dealing with beliefs in mathematics education addresses some perspectives on mathematical concepts or activities, decides which dimensions of beliefs are in scope of the work, or in which ways bearers of beliefs may interact based on their beliefs. These are implicitly the researcher's beliefs in their research. The ‘blind spot’ can thus be resolved by looking closely, in this case by reconstructions, similar to the given example by Grigutsch et al. (1998). Specifically, through an interpretative explication of the perspectives addressed by the researchers, which can then result in an indication of researchers' beliefs by an observer – the readers of the research work or the author(s) of the research work themselves.

Now that we have seen that one way of filling the "blank spot" is to reconstruct researchers' beliefs from existing research literature, the question naturally arises whether one can and should also make one's own beliefs as a researcher explicit, which means *constructing* researchers’ beliefs. Of course, every explication can only be done under the limitations already described. But addressing one's own beliefs can of course be realized in a similar way Grigutsch et al. (1998) have done in their article. So, one's own beliefs can and should be made explicit as well as the presumed limits of these beliefs. Still, another problem might be drawing a distinction between beliefs and knowledge (Pehkonen & Pietilä, 2003).

Despite of that, it still seems as if mathematics education researchers' beliefs are still not recognized widely in the mathematics education literature. I do not think this is due to a lack of reflection in the community, but rather that a culture of openness and appreciation of different beliefs should prevail as well as being visible in research articles. Balacheff (2008) develops an interesting scientific program that addresses these aspects. His starting point is the notion of proof in the mathematics education community. After a detailed reconstruction of different perspectives on proving and mathematical proofs he concludes the diversity in this field. A central part of his scientific program lays in the “elicitation of theoretical commonalities and divergences, and possibly turn them into questions" (Balacheff, 2008, p. 511). This idea can be effective through continuous asking and answering reflective impulse questions during the research process on the investigated objects of belief from the perspective of the researchers–in case of Balacheff (2008)–beliefs about proofs. Specifically, the researcher assigns beliefs to him-/herself by this constructive process and opens another level of scientific discourse.

## 4 How to utilize mathematics (education) researcher's beliefs: Research vs. teaching (an explicit approach)

On a theoretical level it is interesting to think about (mathematics (education)) researchers' perspective, for example regarding differences and similarities to other bearers of belief. Still, it is in question, how these considerations can lead to scientific progress in mathematics education. I belief an answer to this question needs to address the utilization aspect of (mathematics (education)) researcher's beliefs.

In this context, the following uses of reflected beliefs in the research process seem to be particularly relevant:

- To raise awareness of one's own beliefs about the investigated belief object:
    - explicating the limits of one's own research perspectives on these belief objects, as Grigutsch et al. (1998) did in their focus on four aspects of mathematical worldviews,
    - identifying hastily assumed commonalities or differences of different approaches, as Networking of theories as research practice allows (Bikner-Ahsbahs & Prediger, 2014),
    - overcoming mistakenly deadlocked beliefs about concrete belief objects, as Kolmogorov (1956) did, for example, through his construction of a formal-abstract concept of probability,
- to become aware of one's own beliefs about ways of constructing beliefs,
    - deciding whether the chosen methods or timeframes of research are adequate regarding the investigated belief object, e.g., the change of beliefs (Stoffels, 2020),
    - documenting and reflecting one's own development of beliefs and making it become one's own paradigmatic example for belief changes (Altrichter & Holly, 2005),

- to become aware of one's own differentiation between one's own beliefs as (mathematics (education)) researcher and beliefs of other belief bearers:
    - evaluating if and which differences might exist to beliefs of other bearers (Törner, 2018),
    - Evaluating if one has too high or too low expectations towards mathematical learners and teachers, who of course have had different experiences and to whom correspondingly different beliefs can be assigned (Törner, 2018).

Before I discuss the use of (mathematics (education)) researchers' beliefs in teaching I will at this point explicate one of my own beliefs about (mathematics (education)) researchers' beliefs: (mathematics (education)) researchers' beliefs do not differ principally from non (mathematics (education)) researchers' beliefs.

This belief may originate directly from the conceptualization of beliefs described in the first section based on the DSE approach by Bauersfeld (1983, 1985). This means, that the research findings and recommendations for teaching regarding teachers' beliefs may be transferred directly. However, a distinction could possibly lay in the beliefs of how mathematics education can or should be learned or taught. This, admittedly, is a field that has hardly been considered so far, but in which important questions about beliefs of mathematics education presumably arise. Not only regarding mathematics and its teaching and learning, but also regarding their own discipline. Such a perspective on mathematics education can be used, for example, to organize mathematics educational knowledge by explicating beliefs, e.g., for the teaching and learning of calculus (Dilling, Stoffels & Witzke, 2024).

With these preliminary remarks in mind, I would like to emphasize the following uses of mathematics education researchers' beliefs:

- to enable discourses in teaching and to reveal the discourse basis on the teacher side,
- to stimulate multiple perspectives on mathematics, mathematical objects as well as mathematics education,
- to reveal reasons for ways of working in mathematics education, and last but not least,
- to be a role model for learners in making them aware of their own beliefs.

## 5 Final remarks

While writing this article once again I realized how difficult it is to become aware of one's own beliefs–in this case about the beliefs object “(mathematics (education)) researchers' beliefs”–and to be willing to bring them up for discussion.

For me still, the most striking example of this difficulty and inner conflict, which arises in such an undertaking of addressing one's own beliefs, can be found in Kolmogorov's (1956) "Foundation of probability theory". Kolmogorov explicates his view on mathematics by a comment to the reader in his footnotes belonging to paragraphs “§1 Axioms” [2] and “§2 The relation to experimental data” [1] :

> “ [2] The reader who wishes from the outset to give a concrete meaning to the following axioms, is referred to §2.” (Kolmogorov, 1956, p. 2)

> “ [1] The reader who is interested in the purely mathematical development of the theory only need not read this section, since the work following is based only upon the axioms in §1 and make no use of the present discussion. [...]” (Kolmogorov, 1956, p. 3)

Kolmogorov's inner struggle offering a formal formulation of probability theory can be seen in footnote 2, where he offers the reader a concrete interpretation in an empirical context. This is somehow in conflict with his objective formulating a formal-abstract foundation of probability theory, which he states in footnote 1.

I hope this 'how-to' guide to (mathematics (education)) researcher's beliefs may be helpful to focus on this ‘blind spot’ and not to lose sight of it in the future.

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# Characterizing students' beliefs about mathematics as a discipline

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To fully possess mathematical competence and to understand its relevance, importance and aesthetics, it is essential to be aware of aspects of mathematics not only as a school subject but also as a scientific discipline. In a systematic literature review, the theoretical characterization of compulsory school students' beliefs about mathematics as a discipline is investigated, as well as the empirical tendencies in the nature of their actual beliefs. Furthermore, the valuation of these beliefs is addressed. The 18 included studies demonstrate a clear pattern in applying a dualistic/relativistic spectrum when characterizing and analysing students' beliefs about mathematics as a discipline, with students generally possessing dualistic beliefs, which is in contrast to what is favourable to their learning.

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## 1 Introduction

Mathematics is part of the education of all compulsory school students around the world. To be mathematically competent can be defined in many ways, but it is widely studied and generally agreed among researchers of mathematics education that the way in which students perceive the subject is an important factor for their motivation, their learning process, and their approach to mathematical problems etc. (e.g., Furinghetti & Pehkonen, 2002; McDonough & Sullivan, 2014). Several definitions of students' mathematics-related beliefs have been presented in existing literature (e.g., Underhill, 1988; Kloosterman, 1996; Op't Eynde et al., 2002), but not all of these definitions include the dimension of mathematics that exceeds the school subject (e.g., Op't Eynde et al., 2002). However, to fully possess mathematical competence and to understand its relevance, importance and aesthetics, it is essential to be aware of aspects of mathematics not only as a school subject but also as a *scientific discipline*, as pointed out by Niss & Højgaard (2011). The latter could also be characterized as the *nature* of mathematics, and includes the role of mathematics in the world, the development of mathematics, the methods used by mathematicians and the philosophy of mathematics, to name a few examples. Skemp (1976) already noticed the importance of students' beliefs about mathematics as a discipline in his distinction between instrumental and relational understanding, as did Schoenfeld (1985) with the introduction of the term "mathematical world view". Making students aware of mathematics in the world and as a discipline can both provide justification for the school subject and a sense of relevance as well as gives the subject a meaningful context. All of which may increase their motivation and benefit their learning.

As part of a larger PhD project aiming to develop students' beliefs about mathematics as a discipline, I wish to form an understanding of how students' beliefs about mathematics as a discipline or the nature of mathematics have been described and characterized in existing research. In this paper, I therefore approach the subject through a systematics literature review, serving two purposes: 1) to provide information of how students' beliefs about mathematics as a discipline can be categorized and analyzed, and 2) to detect tendencies in the nature of compulsory school students' beliefs about mathematics as a discipline.

## 2 Method

In order to select relevant literature in a systematic manner, certain criteria have been taken into account:

A. Only studies concerning students in primary and secondary school have been included in the review, as students on tertiary educational levels in many cases will have chosen a certain educational path and thereby have a bias in regard to their interest in mathematics.
B. Although there is a wide representation of studies concerning students' beliefs about mathematics in general, this review is restricted to address students' beliefs about mathematics as a discipline or the nature of mathematics. However, some studies addressing students' beliefs about "what is mathematics?" are included in the review as they cover the essence of beliefs about the nature of mathematics. As "mathematics as a discipline" is an ambiguous term, it might be characterized in several ways, including a variety of content. Thus, studies that cover only parts of mathematics as a discipline (e.g., beliefs about the history of mathematics, the role of mathematics in the world or beliefs about problem-solving) have not been included, as such a search might exclude content that some literature considers part of mathematics as a disci-

pline. Furthermore, this study investigates students' beliefs about mathematics as a discipline as an overall concept, not their beliefs on the individual parts or perspectives on this.

C. Only literature published within the last 20 years have been included in the performed searches under the assumption that previous relevant and important literature will be cited in more recent studies. Hence, some references with several citations in the selected studies, or references appearing to be relevant, have also been assessed and included, if fitting the inclusion criteria.
D. To ensure the validity of the included literature, only peer-reviewed studies have been included.
E. As determined by Furinghetti and Pehkonen (2002), the concept of beliefs is not clearly and somewhat unambiguously defined. Even though the term beliefs is the most commonly used in the resulting studies (15 studies), other terms are used as synonyms: conceptions (in 2 studies), views (in 3 studies), and images (in 1 study). These terms may on the other hand cover aspects that are not relevant to this study, which have been considered during the subsequent screening processes.  An elaboration of how these concepts are interrelated or defined in the studies will not be a part of this paper. To leave room for the actual focus of this paper, they will instead all be considered as similar, if not identical, and quite closely related notions that in essence cover the same phenomenon.

Williams and Leatham (2017, p. 377) define the 20 most important journals in mathematics education. To cover these, I conducted searches in ERIC[1] and in Web of Science[2]. Furthermore, proceedings from the MAVI 16–35, PME 24–43 and CERME 2–11 conferences have been manually searched.

[1] Search string in ERIC: noft((belief* OR view* OR perception* OR conception* OR image* OR understanding*) AND math* AND (discipline OR nature) AND (student* OR pupil* OR child*) AND (school OR primary OR secondary) NOT ("STEM" OR teacher*)) AND la.exact("English") AND PEER(yes) AND pd(>20001231) (April 22, 2021)

[2] Search string in Web of Science TS=((belief* OR view* OR perception* OR conception* OR image* OR understanding*) AND math* AND (discipline OR nature) AND (student* OR pupil* OR child*) AND (school OR primary OR secondary) NOT (STEM OR teacher*)) (April 22, 2021)

## 2.1 Summary of search process

The search in databases resulted in 292 studies imported into the review software Covidence, whereof 7 duplicates were removed. 285 studies were screened against title and abstract using the above-mentioned criteria A through E (Table 1). This resulted in a further exclusion of 275 studies. 10 studies were imported from MAVI proceedings, 5 studies from CERME proceedings and 8 studies from PME proceedings. Hence, a total of 33 studies were full-text screened, leading to the exclusion of 26 studies, whereof 5 were excluded due to criterion A (wrong sample group), 20 due to criterion B (not mathematics as a discipline), and 1 due to criterion B (wrong aspect of affect). Finally, 7 studies were included in this review as well as 11 relevant references cited in the 7 included studies.

**Table 1.** Overview of review process and studies excluded based on inclusion criteria. Exclusions related to criteria C and D tool place in step 1.

| | |
|---|---|
| Step 1: | 292 references imported from databases for screening<br>7 duplicates removed (285 remaining) |
| Step 2: | 285 studies screened against title and abstract<br>275 studies removed (10 remaining) |
| Step 3: | 23 studies imported from conference proceedings for full-text assessment |
| Step 4: | 23 + 10 = 33 studies assessed for full-text eligibility<br>26 studies excluded: 5 (criterion A); 20 (criterion B); 1 (criterion E). (7 remaining) |
| Step 5: | 11 studies included from snowballing |
| Step 6: | In all 18 studies included |

## 2.2 Analysis

The research and findings in the included studies have been analyzed from three perspectives. (1) Characterization of what constitutes beliefs about mathematics as a discipline is synthesized from nine studies. Where some researchers apply existing frameworks or categories to their data, others develop their own framework. Studies that do not present a clear framework, categorization or definition are not included in this perspective. (2) Thirteen of the studies present empirical findings that indicate what kind of beliefs students actually seem to possess. These findings are presented in the second section. (3) Eight of the included studies concern the quality of students' beliefs, strongly indicating that there are beliefs about mathematics that are considered "appropriate", "healthy" or "ideal" – beliefs that are preferable to others and thus should be pursued in the students' learning and their cognitive development. Hence, there are also beliefs that are inappropriate and undesired, generally because they do not support the students' learning, motivation, critical sense etc. This sort of ranking of beliefs is the theme of the third section.

## 3 Results concerning the characterization of beliefs about mathematics as a discipline

The characterization of what actually constitutes and is included in students' beliefs about mathematics as a discipline can be approached in different ways. One option is to present a set of categories or issues that define this form of beliefs, and thus list the content of mathematics as a discipline, as done in three of the included studies. These are presented in the following section. The subsequent section describes eight studies that apply another option, namely to describe the characteristics or quality of a person's beliefs within a spectrum.

### 3.1 Content of beliefs about mathematics as a discipline

Based on existing research, Borasi (1993) categorizes beliefs about mathematics in four categories: two concerning mathematical *activity* (nature and scope), and two concerning mathematical *knowledge* (nature and origin). Grouws (1996) operates with similar categories in his framework for analyzing students' conceptions of mathematics, but with a different definition of the dimensions of mathematical knowledge (*composition*, *structure* and *status*) and mathematical activity (*doing* mathematics and *validating* ideas in mathematics). Furthermore, Grouws (1996) adds the dimensions of *learning* mathematics and the *usefulness* of mathematics.

By including beliefs about the learning of mathematics, Grouws (1996) relates the discipline of mathematics to an educational context. However, in Jankvist's (2015) expansion of Op't Eynde et al.'s (2002) model of students' mathematics-related beliefs (Figure 1), he argues that where the original model concerns beliefs about mathematics in a school setting, the added dimension of mathematics as a discipline concerns issues related to *non-school* settings.

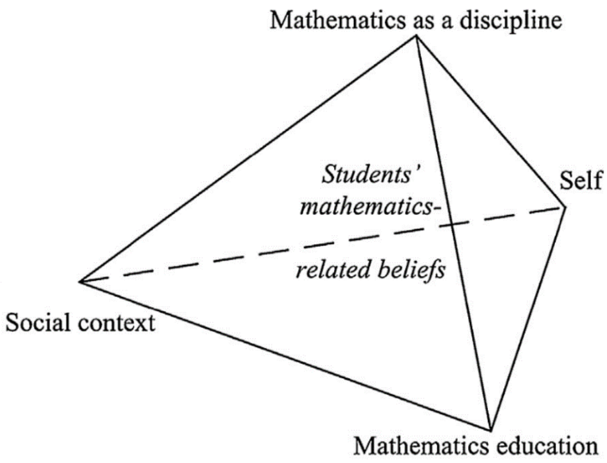


Figure 1. Jankvist's expansion of "Constitutive dimensions of students' mathematics-related belief systems" (Jankvist, 2015, p. 45). The bottom triangle constitutes the original model (Op't Eynde et al., 2002, p. 27).

The dimensions of the original model are (1) beliefs about mathematics education (beliefs about mathematics as a subject, mathematical learning and problem solving and mathematics teaching); (2) beliefs about the self (beliefs about self-efficacy, control, task-value and goal-orientation); and (3) beliefs about the social context, normally the classroom (beliefs about the social norms in the class, i.e., the role and the functioning of the teacher and the role and the functioning of the students aa well as beliefs about the socio-mathematical norms in the class). Included in the fourth dimension concerning mathematics as a discipline are beliefs about mathematics as a *pure science*, an *applied science*, a *system of tools for societal practice* as well as the *philosophical and epistemological nature* of mathematical concepts, theories etc. However, the dimensions of the belief system are interdependent. Beliefs about the issues connected to mathematics as a discipline are to a large degree developed within a school setting and only in the interplay between the three other dimensions. Therefore, the fourth dimension is placed outside the triangle, turning the model into a tetrahedron instead of a square, thus making the three original dimensions the basis on which the fourth is build. Inspired by Spangler (1992), Jankvist further characterizes this category of beliefs through a set of questions (Jankvist, 2015, p. 45):

> [H]ow, when and why mathematics came into being; if mathematics is discovered or invented; where mathematics is applied; if it has greater or lesser impact on society today than previously; if mathematics can become obsolete; what mathematicians do; if mathematics is a scientific discipline.

### 3.2 A spectrum of beliefs

Another way to characterize students' beliefs about mathematics as a discipline concerns the characteristics of the beliefs, or perhaps rather the quality or spectrum of how one perceives the nature of mathematics. This characterization is found in eight of the included studies. The majority presents spectra that range from seeing mathematics as a static, rigid and rule-based discipline, to a dynamic, relativistic and applicable "science of patterns", as described by Schoenfeld (1992, p. 334):

> At one end of the spectrum, mathematical knowledge is seen as a body of facts and procedures dealing with quantities, magnitudes, and forms, and the relationships among them; knowing mathematics is seen as having mastered these facts and procedures. At the other end of the spectrum, mathematics is conceptualized as the "science of patterns," an (almost) empirical discipline closely akin to the sciences in its emphasis on pattern-seeking on the basis of empirical evidence.

A similar spectrum is described in Borasi (1993) and Grouws (1996), who characterize the range of mathematics-related beliefs from *dualistic* to *relativistic*, relying on the framework of Oaks (1989). Each of the aforementioned seven dimensions used by Grouws to define beliefs about mathematics as a discipline is described as a continuum with two poles (Table 2), illustrating the extremes of the spectrum.

**Table 2.** Dimensions for the conceptions of mathematics and their poles on a range from dualistic to relativistic (my extraction from Grouws, 1996).

| **Dualistic** | < ------------------------ > | **Relativistic** |
|---|---|---|
| | 1. composition of mathematical knowledge | |
| facts, formulas and algorithms | < ------------------------ > | concepts, principles and generalizations |
| | 2. structure of mathematical knowledge | |
| collection of isolated pieces | < ------------------------ > | coherent system |
| | 3. status of mathematical knowledge | |
| static entity | < ------------------------ > | dynamic field |
| | 4. doing mathematics | |
| results | < ------------------------ > | sense-making |
| | 5. validating ideas in mathematics | |
| outside authority | < ------------------------ > | logical thought |
| | 6. essence of learning mathematics | |
| memorizing | < ------------------------ > | constructing and understanding |
| | 7. usefulness of mathematics | |
| school subject with little value in life | < ------------------------ > | useful endeavour |

In his study of secondary school students, Grigutsch (1998) also finds that the development of students' views of mathematics can be seen as two contrasting poles (*schema-orientation* (aspects S and RS below) and *process/application-orientation* (aspects P and A)). In essence, these two poles resemble the dualistic and relativistic perspectives, but Grigutsch characterizes the spectrum between them with five different aspects, thereby enabling a more detailed analysis of students' beliefs (Grigutsch, 1998, pp. 174-176):

> F: The Formalism-Aspect (mathematics as logical and precise thinking)
> P: The Process-Aspect (mathematics as a method for considering, understanding and solving problems)
> A: The Application-Aspect (mathematics as useful in daily life)
> S: The Schema-Aspect (mathematics as a collection of rules and procedures)
> RS: The Rigid Schema-Orientation (mathematics is learned (memorized) only to pass exams)

However, the Formalism-Aspect is not easily placed between the dualistic and relativistic poles, and thus Grigutsch's (1998) framework may also be perceived as different aspects that describe students beliefs from a perspective that do not operate within a spectrum.

Gattermann et al. (2012) distinguish between two contrasting categories in their study of students' epistemological beliefs in mathematics, namely *naïve* and *sophisticated* beliefs, the latter being more closely related to deep-processing learning. To measure the students' beliefs, they use a questionnaire composed by items from existing large-scale assessment tools such as PISA and TIMSS. The students' beliefs are assessed within six different conceptual aspects that are closely related to those of Grigutsch (1998). Three of them are related to the category of naïve epistemological beliefs and constitute a conception of mathematics similar to the dualistic view mentioned above: (1) rigid schemes ("exercises in mathematics always have only one right solution"), (2) schematic conception (mathematics as a collection of calculation methods and rules) and realistic conception ("all mathematical problems have already been solved"). Likewise, the aspects related to sophisticated epistemological beliefs resemble the relativistic view: (4) relativistic (mathematics as a coherent system), (5) processes (mathematics can be discovered and constructed by oneself) and (6) relevance/application (relevant for everyday life).

Several of the studies (Gattermann et al., 2012; Grady, 2018; Grevholm, 2011; Halverscheid & Rolka, 2006) rely on a categorization of beliefs (or views) that adds a third category, as presented by Ernest (1989) (instrumentalist, Platonist and problem-solving view) or the corresponding notions of Dionne (1984) (traditional, formalist and constructivist perspective). I shall therefore briefly summarize the essence of Ernest's notions. Parallel to the aforementioned dualistic view of mathematics, the *instrumentalist* view is characterized by perceiving mathematics as "a set of unrelated but utilitarian rules and facts" (Ernest, 1989, p. 249). In the *Platonist* view, mathematics is characterized as "a static but unified body of knowledge" (ibid.), while the *problem-solving* view, similar to the relativistic view, characterizes mathematics as "a dynamic, continually expanding field of human creation and invention" (ibid.). As was the case with the framework presented by Grigutsch (1998), Ernest's notions are not necessarily defined within a spectrum, but rather as a characterization of three different perspectives on mathematics that are not opposites.

An alternative approach to students' beliefs are introduced by Grady (2018). In her study, she presents a framework for describing and analyzing students' enacted conceptions of the nature of mathematics from their behaviour instead of the often used self-report data. From behavioural indicators, the framework can be used to assess to what degree students conceive mathematics as *sensible*, which is defined as viewing mathematics as "a coherent, connected system that can be reasoned about and used to describe and reason about the world at large" (Grady, 2018, p. 127). As the reader might notice, this definition has common features with both Oaks' relativistic view, Grigutsch's process/application-orientation and Ernest's problem-solving view. The degree to which students' hold such a conception is assessed based on four dimensions of behaviour (as well as the students actually stating that mathematics makes sense): 1. strategizing (e.g., discussing methods or seeking alternative solutions), 2. expecting explanations (e.g., justifying, reasoning and inquiring), 3. expecting/seeking connections (within mathematics and to other contexts), and 4. assuming authority (e.g., inventing problems of their own or checking answer with an alternative strategy). The framework thereby contributes to an understanding of the action-oriented dimensions of students' conceptions of the nature of mathematics.

Having established several frameworks for characterizing students' beliefs about mathematics as a discipline, next step is investigating what kind of beliefs students' actually hold when studied empirically, both in relation to the frameworks as well as in the form of concrete examples of students' beliefs.

### 3.3 Results concerning students' actual beliefs about mathematics as a discipline

Based on experience, discussions with teachers and students (Garofalo, 1989) and existing research (Schoenfeld, 1992), the selected literature offers concrete examples of beliefs about the nature of mathematics that typically are held by students. One of them is that mathematical problems only have one correct answer and that it can only be solved using the correct rule, formula or procedure, usually shown by the teacher. Thereby mathematics will most likely be perceived as a fragmented set of rules and formulas that must be memorized and applied appropriately. This is connected to another typical belief related to the nature of mathematics: that mathematics is not created by "ordinary" people, but must be transferred (normally from teacher or textbook to student) and memorized. Hence, students feel unable to produce mathematics on their own (Garofalo, 1989), and a deeper understanding of the rules and formulas becomes irrelevant, as does formal proof (Schoenfeld, 1992). Furthermore, mathematics is typically believed to have little relevance to the real world but is merely seen as a school subject (ibid.). These are generally beliefs that can be said to reflect what in the previous section is characterized as a dualistic perspective, which is confirmed by Underhill (1988) in his review of mathematics learners' beliefs. In general, students at all ages emphasize memorization and algorithms as important in mathematics, which foster what Skemp (1976) categorizes as instrumental learning, not relational understanding.

Regarding empirical findings, the majority of the studies included in this review, present results that confirm such a tendency. For example, Halverscheid and Rolka (2006) find that half of 28 fifth grade students hold an instrumentalist view of mathematics. Kloosterman (1996, 2002), Grootenboer (2003) Grevholm (2011) all find that students' beliefs about mathematics as a discipline are generally linked and maybe even limited to numbers and calculations, although there are indications that students do not seem to have given much consideration to mathematics as a discipline (Kloosterman, 2002, Grevholm, 2011).

Even though most research thus points to students beliefs about mathematics as a discipline being rather traditional/instrumental/dualistic, some of the included studies show a slightly more complex picture of students' beliefs.  McDonough (1998) shows in her in-depth study of two third grade students' engagement in mathematical procedures that students' beliefs about the nature of mathematics might be more complex, subtle and broad than reported in other research studies. In ten one-to-one interviews with each student over a period over five months, the nature of mathematics is discussed through e.g., photographs of both school and non-school activities, personal definitions for mathematics and ending the sentence “Math is like…”. In both cases, what first appears to be a simple and clear classification of beliefs turns out to be quite complex and ambiguous during the analysis of the data collected. For example, numbers initially appear significant for one of the students, but during subsequent discussions, it becomes clear that she puts more emphasis on measurement and estimating and mainly relates mathematics to non-school activities.

Gattermann et al. (2012) find in their aforementioned study that the 145 secondary school students in average score relatively high on scales addressing the sophisticated epistemological beliefs. Their scores on naïve beliefs are proportionally low. However, the contrary applies to the aspects of relativistic conception (low score) and schematic conception (high score). Thereby, these students do not perceive mathematics as a coherent system, but rather a collection of exact methods and rules for calculation, even though they find mathematics process-oriented and useful in daily life.

According to Schoenfeld (1989), 230 mathematics students in grades 10-12 hold apparently contradicting beliefs about mathematics as a discipline. In their responses to a questionnaire concerning their mathematics-related beliefs, including their view on mathematics as a discipline, the students for example state that mathematics is a discipline of creativity, logic and discovery, but at the same time emphasize the importance of memorization in the learning of mathematics. The students generally separate school mathematics from abstract mathematics, and Schoenfeld suggests that the reason might be that the students’ behavior is driven by their experiences with mathematics rather than what they are told or what they value as “appropriate” beliefs. Likewise, both Schoenfeld (1992) and Garofalo (1989) stress that students to a large extent form their beliefs based on their experiences in the classroom, and that these beliefs thereby is a reflection of how mathematics is presented, performed and evaluated in the educational system. Grootenboer (2003) confirms this by pointing out that the views of students’ in his study are firmly grounded in school experiences.

A noteworthy result is presented in Grouws (1996), who compares the conceptions of mathematics of 55 talented and 112 average high school students. Here, he finds that while the two groups generally see mathematics as a dynamic and useful field, there are noteworthy differences in their conception of doing and learning mathematics. Where the average students – as in Gattermann et al. (2012) – largely follows the above described dualistic way of perceiving mathematics as a discrete system of facts and procedures based on memorization, the talented students view mathematics from a more relativistic perspective. They tend to see it as “a field composed of a system of coherent and interrelated concepts and principles, which is continuously growing. Doing mathematics is a sense-making process in which one must rely on personal thought and reflection to establish the validity of that knowledge” (Grouws, 1996, p. 31). A corresponding result is found by Grigutsch (1998) where the process/application-orientation is increasingly significant among 12th grade students in the high-performance class, compared to the students in the basic level class, who tend to have a more schema-oriented view of mathematics. In lower grades involved in the study (grade 6 and 9), the two poles of the beliefs spectrum are not as distinct, and the students beliefs seem to be more of a mix of the five different aspects in Grigutsch’s framework (cf. the previous section). Both of these studies indicate that certain beliefs might be related to high performance in mathematics and thus are preferable and worth striving for. This issue is unfolded in the following section.

## 4 Are some beliefs better than others?

The overall impression from the studies included in this review is quite clear when it comes to what sort of beliefs to aim for in the teaching of mathematics, and which are considered unfavourable. In relation to the spectrum of beliefs, there seems to be consensus that beliefs belonging to the dualistic pole are not conducive for students’ leaning, motivation or self-concept. According to Borasi (1993) and Schoenfeld (1992), such beliefs impoverish mathematics and do not reflect its nature. In contrast, the relativistic end of the spectrum is seen as unambiguously beneficial. For example, Gattermann et al. (2012) finds that sophisticated epistemological beliefs are more related to a higher degree of self-concept and performance compared to naïve beliefs. This corresponds with the results of Grouws (1996), who as mentioned relates the relativistic perspective to talented students, and Grigutsch (1998), who connects the process/application-orientation to both high performance, motivation and a positive self-concept in mathematics.

Spangler (1992) presents 11 open-ended questions aimed both at assessing students’ beliefs about mathematics on the one hand, and at making the students aware of own their own beliefs. These questions indirectly indicate that the ideal beliefs belong to the relativistic end of the spectrum. For example, students are encouraged to consider the possibility that different answers to the same mathematical problem can be equally correct, that mathematics is more than memorization or computation, and that mathematics is used in many non-school situations, fields and careers. The behavioural indicators in the framework presented by Grady (2018) illustrate what kind of behaviour that is connected to such a perception (see previous section).

The ideal beliefs about mathematics as a discipline are in Jankvist (2015) presented as beliefs that are held evidentially (cf. Green, 1971), i.e., beliefs supported by evidence from examples, experience, reasoning etc. Evidentially held beliefs are more likely to be changed with reason or through reflection. Thus, mathematics education should aim for developing students' *reflected* image of mathematics as a discipline by providing opportunities for experiences and reflection. In a didactical perspective, Jankvist (2015) relates three types of mathematical overview and judgment to the development of students' beliefs. The three forms of overview and judgment are described in the Danish mathematics competencies framework, the so-called KOM-report (Niss & Højgaard, 2011, 2019), and concern: (1) the actual application of mathematics in other subject and practice areas; (2) the historical evolution of mathematics, internally as well as in societal context; (3) the nature of mathematics as a subject. These have a certain equivalence to one of the visionary aims set up by Ernest (2015) for school mathematics with an intention to "contribute to students' mathematical confidence, mathematical creativity, social empowerment and broader appreciation of mathematics" (Ernest, 2015, p. 189). Especially the latter of these aims (broader appreciation of mathematics) is related to the students' beliefs about the nature of mathematics and requires an increased awareness of the following aspects (p. 191):

- mathematics as a central element of culture, art and life, present and past, which permeates and underpins science, technology and all aspects of human culture.
- the historical development of mathematics, the social contexts of the origins of mathematical concepts, its symbolism, theories and problems.
- mathematics as a unique discipline, with its central branches and concepts as well as their interconnections, interdependencies, and the overall unity of mathematics.
- the way mathematical knowledge is established and validated through proof [...], as well the limitations of proof.
- a qualitative and intuitive understanding of many of the big ideas of mathematics (pattern, symmetry, structure, proof etc.)

Where the first aspect can be seen as a parallel to the first form of overview and judgment concerning the application of mathematics, Ernest's second aspect resembles overview and judgment about the historical evolution of mathematics. The last three aspects are all included in the third form of overview and judgment concerning the nature of mathematics as a subject.

## 5 Concluding remarks

The review of the literature shows a clear pattern in the research on students' beliefs about mathematics as a discipline. Regarding the first purpose of this study—to provide information of how students' beliefs about mathematics as a discipline can be categorized and analyzed — the majority of the included studies place this dimension in a non-school setting, addressing aspects of mathematics in the "real world". The characterization of these beliefs overall relies on a more or less nuanced version of the dualistic/relativistic framework, spanning from perceiving mathematics as a static body of facts and procedures to be memorized, to viewing it as a dynamic, coherent, sense-making system that plays an important role in the world and in life. The second purpose regarded tendencies in the nature of compulsory school students' beliefs about mathematics as a discipline. Here, the findings in the included studies show that students in general tend to possess beliefs belonging to the dualistic end of the spectrum, with an emphasis on numbers, calculations and memorization.

As more than one researcher underline, is worth noting that students largely base their belief on their experiences in the classroom, and the results of this review indicate that quite few of these experiences include aspects connected to mathematics as a discipline. Consequently, it must be considered and taken into account which beliefs are favourable to students' learning and appreciation of mathematics. Again, the literature is quite clear in their recommendation of beliefs belonging to the relativistic end of the spectrum. In conclusion, a comprehensive change in students' beliefs is required, ensuring that they are based on experiences that represent mathematics in a more relativistic perspective as well as on evidence and reflection.

It is striking, how relatively few studies were found concerning primary and secondary students' beliefs about mathematics as a discipline in the search for literature. The reason for this might be found in the search strategy. Broader criteria for age group, time span or object (e.g. mathematics in general) might have led to a higher number of relevant hits. On the other hand, as seen in the analysis, mathematics as a discipline can be characterized in multiple ways and with various content. A search strategy addressing the individual issues might also lead to an increased base of results. Nevertheless, the reasons for the apparently low interest in the field must be considered, especially because of the importance of the students' beliefs to their learning and educational well-being as well as the contrast in their actual beliefs and the beliefs considered ideal.

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# Investigating the complex relations among affective variables in the context of gambling

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Gambling disorder is a dramatic phenomenon that is spreading, in Italy as well as around the world, among younger and younger people every year. Activities in mathematics lessons at school can help pre-vent it, but it is necessary to know with which attitudes and beliefs students approach such mathematics lessons, as well as the role of the social environment. Thus, in this study, within a sample of secondary school students who experienced gambling at various levels of addiction (from none to high), we investigate the role of: mathematics-related beliefs, emotions, social relationships, attitudes towards gambling and behaviour, through a set of calibrated self-report multiple-choice questionnaires. This represents for us an opportunity to understand the complex relations among affective variables in mathematics educational activities aimed at preventing gambling disorder. For example, we found a positive correlation between mathematics-related beliefs and gambling frequency, and a negative correlation between emotional regulation and gambling frequency. Hence, we can say that affective variables such as emotions and beliefs have an effect on gambling behavior.

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## 1 Affect as a system

As, in the opening of his contribution to the Mathematical Views conference that took place in 2017, Liljedahl (2018) notes, affect-related research has known for a long time that affective variables influence each other. Hovewer, Peter Liljedahl interestingly argues also that “Research in the affective domain has often been restricted to focused attention on a single affective variable. This is ironic given that we know that affective variables tend to cluster. Perhaps the reason for this is that we lack theories for thinking about affective clusters” (Liljedahl, 2018, p. 21). Liljedahl’s paper, thus, proposes to consider affect as a constellation of beliefs, attitudes, emotions, goals and efficacy. Each person holds their own affective system, which can be represented using a connected graph. Connections among the affective variables, which are the nodes of the graph, are established by the either quasi-logical (among beliefs and conceptions), or psychological (among the other variables) organisation the individual feels with respect to mathematics, its learning and its teaching. That is to say, an individual’s experience, as well as their affective reactions to experience, affect the way their affective system is organised.

In the very last years, some attempts have been made in order to understand affect as a system as a research *problematique* (see Liljedahl, 2018, and references therein). Our study, which consists in the analysis of statistical correlations among responses given to multiple choice questionnaires, previously calibrated, aims at contributing to this research goal by considering a number of affective variables, which we firstly outline separately in the next section and, in doing so, we also recall the theoretical foundations of the questionnaires used in order to conduct the research.

## 2 Affective variables

### 2.1 Beliefs and conceptions

According to Furinghetti and Pehkonen (2002), beliefs are the conclusions that an individual draws from their perceptions and experiences in the world around them. Beliefs can be understood as subjective knowledge: they are propositions about a certain topic that are regarded as true (Philipp, 2007). Being continuously subject to new experiences, beliefs can change, and new beliefs can be adopted (Furinghetti & Pehkonen, 2002). When a new belief emerges, it never comes in isolation from other beliefs, but becomes part of, what has been called, an individual's belief system. According to Green (1971), in fact, beliefs tend to form clusters, as they "come always in sets or groups, never in complete independence of one another" (Green, 1971, p. 41). These clusters form a system, which is organised according to the quasi-logical relations between the beliefs and the psychological strengths with which each belief is held (Green, 1971). Belief clusters are, thus, almost coherent families of beliefs across multiple contexts: for example, beliefs about the nature of mathematics and about its learning tend to cluster in a quite coherent way, for a student. This has probably led Furinghetti and Pehkonen (2002, p. 41) conclude that "an individual's conception of mathematics [is] a set of certain beliefs" namely to understand conceptions as clusters of beliefs.

The Gambling Belief Questionnaire (GBQ), developed by Steenbergh, Meyers, May & Whelan (2002), conceives beliefs and conceptions as rather synonymous. This is in line with Furinghetti and Pehkonen's (2002) understanding of the relation among the two. What is interesting for our own study is the characterisation of conceptions/beliefs according to the degree of intersubjective consensus and of subjection to disputes, which distinguishes them from knowledge and which allow us to talk about mis-conceptions when subjective knowledge is in contrast with mathematical knowledge. We, thus, understand misconceptions as cluster of beliefs that are in contrast to mathematical knowledge. In the context of gambling, some important misconceptions play a crucial role, as they cause individuals to overestimate their level of control over the outcome of the game and diminish the role of chance (Philander, Gainsbury & Grattan, 2019). Research on gamblers' cognitive distortions suggests that they are an important component to understand both normative and disordered gambling behavior (Philander et al., 2019): evidence suggests that cognitive distortions lead to continued gambling despite significant financial loss and play a causal role in the maintenance and development of gambling disorders (Philander et al., 2019). Mathematical affect-related research has provided evidence that beliefs have observable behavioural consequences (e.g. Di Martino & Zan, 2011), and this can have dramatic consequences when students develop beliefs about mathematics that lead them to choose a career path that avoids it.

## 2.2 Attitudes

Di Martino and Zan's (2011) approach to attitudes in mathematics fits within the investigation about students' attitudes towards gambling (Ferris & Wynne, 2001). Di Martino & Zan, in fact, focus on the links among three dimensions that characterise their model of attitude, namely: (i) emotional disposition towards mathematics; (ii) vision of mathematics; and (iii) perceived competence in mathematics. Di Martino and Zan also argue that the relations among the three dimensions turn out to be "causal not in a logical sense but rather in a social, ethical and psychological one" (p. 480).

Within our study, we adopted the European School Survey Project on Alcohol and other Drugs (ESPAD). Interviewees are asked to rate their positive emotional disposition towards a list of betting games, as well as their adversity towards them, which can be understood in terms of emotional dispositions towards gambling. Moreover, the Canadian Problem Gambling Index (CPGI), whose development was associated to a rejection of a medicalised model of pathological gambling in favour of a view of problem gambling as a social issue (Ferris & Wynne, 2001), asks interviewees to state whether in the past 12 months it happened to them to bet more money than they wish, or to spend more time than they planned, on gambling. As beliefs about one own's ability to win betting games is assessed by GBQ, it is possible to investigate the relation between an individual's misconceptions about gambling and its outcomes, their appreciation/

aversion towards betting games, their perceived competence on gambling and the problematic behaviour of gamblers. To note, the construct of attitude, in the way it is defined by DiMartino and Zan, includes self-efficacy, which is a central variable in the context of our research.

## 2.3 Emotions

In affect-related research, emotions are deemed as the most intense and unstable, as well as the least cognitive-related, dimension of affect (Hannula, 2012). Di Martino, Gómez-Chacón, Liljedhal, Morselli, Pantziara and Schukajlow (2015) annotate that emotions are recently gaining increasing attention, given the emerging acknowledgement of the role that emotions have in learning. With Goldin (2000), we maintain those emotions play an important role in human coping and adaptation. In Goldin's study, focused on problem solving, the solver experiences fluctuating affective states and they can exploit them during problem solving, in order to store and provide useful information, facilitate monitoring, and evoke heuristic processes.

This viewpoint is particularly relevant in the context of gambling, given the lack of emotional regulation that often characterises problematic gambling behaviour. According to Balzarotti, John & Gross (2010), in fact, emotion regulation is crucial not only in this context, but generally in various aspects of healthy adaptation, from affective functioning to social relations. In particular, two emotion regulation strategies have been paid attention to, namely: cognitive reappraisal, which consists of attempts to think about the situation so as to alter its meaning and emotional impact, and expressive suppression, which consists of attempts to inhibit or reduce ongoing emotion-expressive behaviour. Analysing how emotions unfold over time, and particularly interesting in the context of gambling, it has been argued that reappraisal and suppression have their primary impact at different points of the emotion-generative process (Balzarotti et al., 2010): while reappraisal is a strategy that is activated before the complete emotional response has taken place and is, thus, expected to modify the entire temporal course of it, suppression is a response-focused strategy that intervenes once an emotion is already under way and after emotional responses have already been fully generated. It thus might be expected to require repeated efforts to manage emotional responses as they continually arise, taxing the individual's resources (Balzarotti et al., 2010).

The Emotion Regulation Questionnaire by Balzarotti et al. (2010) measures such aspects of emotional regulation, which is theorised to have an impact on an individual actions by affecting behaviour, but also on an individual's reflexive interpretation of their own actions. In other words, emotional regulation is connected to an individual's interpretation of their experience, and thus to beliefs and attitudes.

### 2.4 Social relations (parents)

Within the MAVI community, it has been observed by Natascha Albersmann and Marc Bosse that "parental influences on their children's mathematical developments are especially notable in direct supportive situations, like homework situations, in which parents and their children get in contact with mathematics in an active way" (Albersmann & Bosse, 2016, p. 163). All in all, in any situation (even outside mathematics educational contexts), in which students and their parents enter in relationship, the latter tend to affect the way the former perceives and interprets what is going on —as a matter of fact. Thus, on one's side, parents' (mathematical yet pedagogical) knowledge turns out to be important in order to understand a child's attitude and beliefs towards it. On the other side, the kind of parental support a student receives can make a difference on the way they organise their belief system. For example, Albersmann and Bosse (2016, p.163) note:

> "Parents' involvement is particularly beneficial for children when it is, for example, autonomy supportive, focuses on the process of learning, and is accompanied by positive affect. However, it has negative repercussions for children if the involvement is controlling, performance focused, and accompanied by negative affect."

In the context of gambling, we explore if and how different kinds of parental support at emotional, relational and cognitive levels is related to students' attitudes and their mathematics-related beliefs towards gambling. This is investigated through the European School Survey Project on Alcohol and other Drugs (ESPAD).

### 2.5 A systemic view of affective variables

Inspired by the systemic approach to beliefs as tending to cluster, Liljedahl (2018) extends the metaphor of belief system to comprise the whole set of affective variables as a system. Namely, beliefs, attitudes, emotions form a system and are as well intertwined with experiences, needs, goals and personal relationships that frame an individual's life. As Liljedahl (2018, p.24) exemplifies:

> "Consider, for example, a student who has low self-efficacy about doing mathematics. This student would also, likely, have high anxiety around writing mathematics tests. In this relationship we could say that the anxiety is a logical derivative of the primary affective variable of low self-efficacy (affect–affect causality). But both low self-efficacy and high anxiety may have actually arisen jointly from some negative experience involving poor performance on a test accompanied by some sort of negative consequence like being scolded by a parent or some form of public shaming (experience–affect causality). The reality is, however, that once established, whether it is derived from a negative test experience or directly from low self-efficacy, this student's anxiety will quickly become a robust affective variable on its own right. As such, within this framework, and for the purposes of the research presented here, affect–affect relationships are considered to not have a primary-derivative relationship [...]. However, environment–affect relationships are seen as causal [...]"

This model theorises the relations among the affective variables, but at the present, up to our knowledge, a systematic and quantitative analysis of the nature of the relations among them is missing within the research field. Our study aims at contributing to this goal. We, thus, draw on this model to both explore relationships among affective variables in the context of gambling, and contribute to describe the systemic nature of affect, by seeing how each variable influences the other and how the system as a whole is organised.

## 3 Methodology

We propose a systemic, interconnected approach to affective variables and we answer the following research questions: is there a correlation among different affective variables in the context of gambling? What is the nature of such a correlation?

### 3.1 Participants

For the research, 1218 students have been interviewed. They come from 4 secondary schools in the Northwest of Italy, which voluntarily adhered to the project: 328 of them are from the Lagrangia Higher Institute in Vercelli, 279 from the CNOS-FAP Rebaudengo, 321 from the Agnelli Higher Institute and 290 from the CNOS-FAP Valdocco. A diversity of mathematical achievement, socio-economic status and betting frequency among the students has been searched during the sampling phase. We underline that the sample under study was interesting not solely with regard to its numerosity. The students involved in the research come from neighbourhoods that are peculiar: they live in truly multiethnic environments, with a variety of cultures, socio-economic statuses and conceptions that make it interesting to investigate, especially because no direct relations between differences in such dimensions and behaviour/beliefs with respect to gambling emerged. Participants include 788 males and 430 females, with an age range of 14-19 years (mean age 16.1 ± 1.3 years). Namely, these students attended grades from 9 to 12. They participated on a quasi-voluntary basis, that is to say that their teachers were contacted and, if they accepted, their respective classes were administered the questionnaires.

### 3.2 Data gathering

Questionnaires have been administered at school, on an online platform and anonymously. The following questionnaires have been proposed to students:

1. the Emotion Regulation Questionnaire (ERQ) (Balzarotti et al., 2010), a self-report questionnaire consisting of 10 items with responses on a 7-point Likert scale, ranging from 1 = "Strongly disagree" to 7 = "Strongly agree". It is composed of two scales that correspond to two different emotion regulation strategies: cognitive reappraisal and expressive suppression. The reliability was good for both the reappraisal scale (α = 0.84) and the suppression one (α = 0.72). Sample questions are: “When I feel bad, I try to look at things from a different perspective”, “When I face a difficult situation, I try to see it under a light that allows me to keep calm”. The Italian translation was developed for this project by the IUSTO psychological research area with a back-translation procedure.
2. Canadian Problem Gambling Index (CPGI) (Ferris & Wynne, 2001), a self-report tool that allows the researcher to identify any behaviour related to excessive or pathological gambling. It is made up of 9 items on a Likert scale from 0 = "Never" to 3 = "Almost always". The questionnaire has a good internal consistency (α = 0.87).
3. Gambling Belief Questionnaire (GBQ) (Steenbergh et al., 2002), a self-report questionnaire that measures cognitive distortions related to gambling, through 21 items with responses on a 7-point Likert scale, from 1 = "Strongly disagree" to 7 = "Strongly in agreement". In the GBQ questionnaire, we can single out two items that exemplify mathematical misconceptions in the context of gambling, that are: (A) “if I am gambling and I am losing, I have to

continue because I do not want to lose the opportunity to win" and (B) "I have a lucky strategy that I use when I gamble". Items labelled with A and B highlight two beliefs about gambling which conceal two misconceptions in probability, which regard the probabilistic independence of two events. The questionnaire has a really good internal consistency (α = 0.94).

4. European School Survey Project on Alcohol and other Drugs (ESPAD), a self-report questionnaire that opens with a series of questions aimed at framing the interviewees' socio-cultural context and then investigates the consumption of legal substances such as tobacco, alcohol, psychotropic drugs, doping and other not-legal psychotropic substances, as well as betting games (which are not-legal for people younger than 18 years in Italy). Specifically, a distinction is made between the experiences of use of substances in general (within one own's life period), in the last 12 months and in the last 30 days. Attitudes of approval/disapproval with respect to the use of the various substances and the perception of the risks related to them are finally asked. In the ESPAD questionnaire, students were asked to rate their agreement with respect to statements like: (1) "My parents know the places where I go when I am not at home"; (2) "I feel welcomed by my parents"; (3) "I easily receive money from my parents when I ask".

Students took around 40 minutes to respond to the entire set of questionnaires.

### 3.3 Method of data analysis

In order to answer our questions about the existence and the nature of the correlation among different affective variables in the context of gambling, multivariate statistical analyses have been conducted. In particular, linear correlation reveals the extent to which variables are related to each other, in terms of how a variable increases (e.g. positive attitude) when another variable increases (e.g. enjoyment). Multiple linear correlation considers a set of (independent, labelled $x_i$) variables as linearly related to a (dependent, y) one. It is, thus, possible to compute different regression models and to estimate the correlation among the variables of interest. But it is also possible to combine linear regression models so as to investigate whether a variable explains a significant proportion of variance in the dependent variable (y), after accounting for all other variables. Hierarchical regression is the name of this technique.

In our study, we firstly compute the linear correlation among pairs of variables measured by the various questionnaires (e.g. correlation among perceived parental support and frequency of gambling). Then, when correlation was significant, hierarchical regression has been applied so as to check whether variables, for example related to social environment and emotional regulation, play a predictive role with respect to the gambling frequency and mathematics-related beliefs in gambling.

## 4 Results

We start with investigating the correlation between gambling frequency, measured by CPGI, and mathematics-related beliefs about gambling, measured by GBQ. Between the two, there is a rather strong positive correlation ($r=0.384$), which is statistically significant with a p-value lower than 0.001 ($p<0.001$ in the sequel). This means that, the more the misconceptions about gambling, the more the gambling frequency, and the probability to be wrong saying that there is a correlation of about 0.4 between the two is lower than 0.1 %. We can, thus, start representing the system of affective variables related to gambling and mathematics and weighting the link between the act of gambling and beliefs about it, as in Figure 1. Gambling frequency is strongly ($p<0.001$) and positively correlated with mathematics-related beliefs. A strong correlation is represented by a solid line, while a correlation with $p<0.05$ is represented by a dashed line and a weaker one ($p<0.10$) by a dotted line. Labels with 'positive' and 'negative' are attached to the connecting lines.

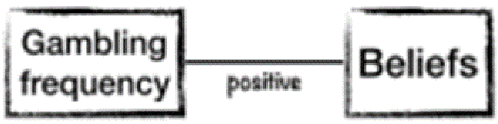


Figure 1. starting to characterise the affective system in the context of gambling, with a focus on mathematics related beliefs about it.

We now examine the systemic nature of the relations between mathematical beliefs/ misconceptions and family environment. Specifically, we explore how family factors (parents' education, student's satisfaction in the relationship with parents, and perceived parental support from ESPAD) were related to beliefs within gambling (from GBQ). Family factors show significant correlations between parents' schooling and mathematical beliefs related to gambling (Table 1). Specifically, there is a statistically significant correlation ($p=0.007$) between mother education and the total GBQ score: r=-0.113 means that the higher the mother education level (from elementary to graduate), the lesser the misconceptions about gambling. There is also a significant (with a *p-value* at level of p=0.05) correlation between items A and/or B and parental support: weaker mathematics-related misconceptions about gambling are positively related to how much parents know the places frequented by the student (item 1 above) and how much a student feels welcomed (item 2). Gambling frequency increases when financial support provided by parents increases (item 3). To note, there is also another item that investigates financial support (i.e. "I receive a fixed amount of money by my parents on a monthly basis"), but the responses to this are correlated neither to gambling frequency nor to beliefs.

On the basis of these preliminary correlations, hierarchical regressions were performed to verify which family variables play a predictive role on the frequency of gambling activity and on beliefs. The results of the hierarchical regression analyses show that parental support explains 4.3 % of the variance of gambling frequency (daily, weekly or monthly). Conversely, parental education accounts only for 0.2 % of the variance in betting rate, and this is not statistically significant. Namely, regardless of the level of mother and/or father instruction, students tend to bet with higher or lower frequency. As regards mathematical beliefs about gambling (exemplified in items A and B), parental support explains 5.5 % of the variance and this effect is significant. In particular, the more strictly the rules the more the misconceptions (r=0.09 with p=0.05). The more the parents know about the places where students go, the lesser the misconceptions (r=-0.246 with p=0.001). The higher the financial support, the more the misconceptions (r=0,103 with p=0.05).

**Table 1.** Correlations among relationship with parents and gambling. Not statistically significant values are not reported.

| | Mother education | Parents know places frequented by student | Student feels welcome by parents | Parents' economic support |
|---|---|---|---|---|
| Gambling frequency (ESPAD) | r = -0.133 (p = 0.007) | | | r = 0.103 p < 0.10 |
| total GBQ score | | | | |
| item A (GBQ) | | | r = -0.078 p < 0.10 | |

| | | | | |
|---|---|---|---|---|
| item B (GBQ) | | r = -0.095<br>p < 0.05 | r = -0.093<br>p < 0.10 | |

On the basis of this information, the affective system depicted in Figure 1 can be updated and Figure 2 can be drawn.

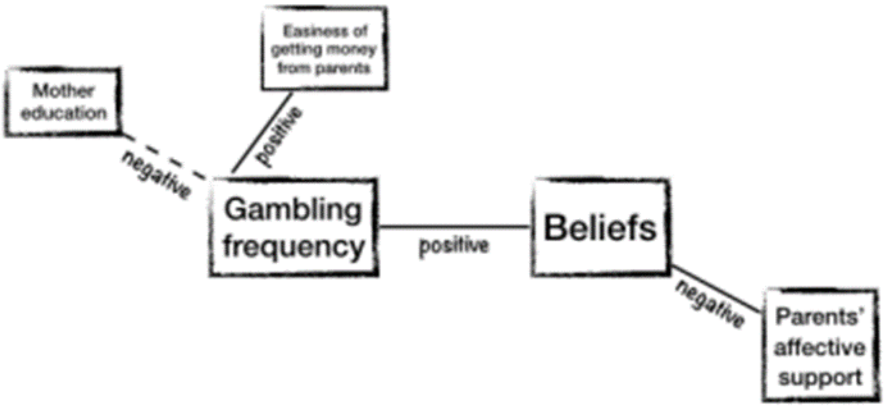


Figure 2. a refinement of the affective system, after the investigation about the role of parents' support with respect to gambling frequency and related beliefs.

We now focus on emotional regulation and mathematical-related beliefs about gambling (Table 2). Expressive suppression from ERQ questionnaire correlates significantly with both gambling frequency, the total score of GBQ questionnaire and items A and B. Correlation indexes in Table 2 mean that the more a student agrees with statements like "When I am happy, I try not to let people notice it", or "I try to control my emotions", the less they bet but the more they hold mathematical beliefs in contrast with probability knowledge. Moreover, cognitive reappraisal from ERQ questionnaire correlates statistically significantly with gambling frequency, as well as with the total score of GBQ questionnaire and item B. Similarly, to emotional suppression case, they are likely to bet less but to hold stronger misconceptions. Hierarchical regression further reveals that emotional regulation explains 7.5 % of the variance of the gambling-related mathematical beliefs, and this is statistically significant.

**Table 2.** Correlations among emotional regulation and gambling. There is a not-statistically significant value, which is not reported.

| | Emotional expressive suppression | Cognitive reappraisal |
|---|---|---|
| Gambling frequency (ESPAD) | r = -0.153<br>p < 0.10 | r = -0.1214<br>p < 0.001 |
| total GBQ score | r = 0.246<br>p < 0.001 | r = 0.168<br>p < 0.001 |
| item A (GBQ) | r=0.189, p<0.001 | |
| item B (GBQ) | r = 0.179<br>p < 0.001 | r = 0.156<br>p < 0.001 |

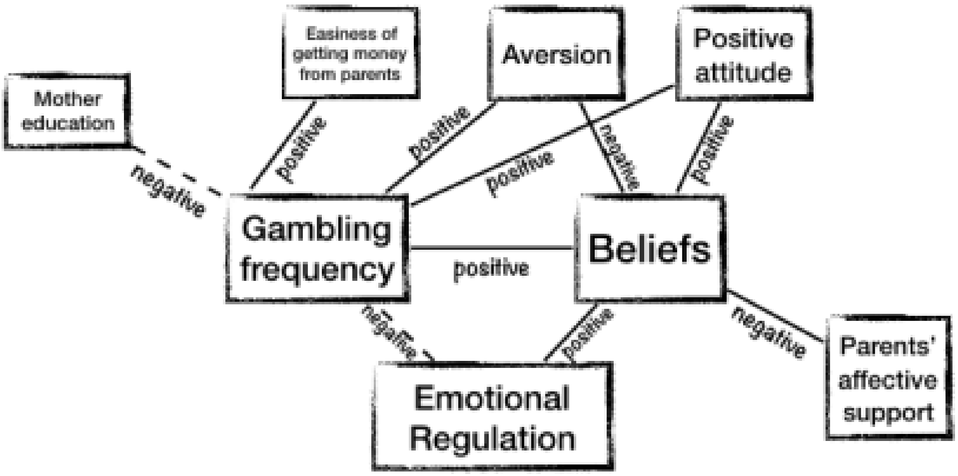


Figure 3. the final affective system in the context of gambling, as it emerges from our study.

Finally, we investigate the role of attitudes towards gambling, and in particular positive ones (i.e. how much a student "likes soccer sport bets", for example) and negative ones (i.e. how much does they think it is hazardous in terms of possible economic loss to bet on soccer plays, for example). As regards the former, both gambling frequency and mathematics-related beliefs increase as positive attitude of betting games increases, with significance p<0.001. As regards the latter, aversion towards risk increases with gambling frequency, but misconceptions decrease with it. In both cases, statistical significance is really good (p<0.001). The two attitudes are also related in an interesting way: negative attitudes towards gambling are higher both for those who like gambling the most, and for those who like it the least, whilst it is significantly lower ($p<0.001$) for those who have intermediate attitudes. On the basis of this further information, the affective system can be featured as in Figure 3.

## 5 Discussion

In this paper, we investigate the correlations among affective variables with respect to students' views and habits about gambling and we focus on mathematics-related beliefs in this context, positive attitudes and aversion, emotional regulation, and family context. It emerges that, among a number of variables related to a student's relationship with their parents, mother's instruction level, parents' affective support and easiness of getting money (by just asking money to one's parents) play a significant role with respect to both gambling frequency and beliefs about gambling. As well, self-declared emotional regulation is correlated with both gambling frequency and beliefs. Also attitudes towards gambling are correlated with the two. Hence, we can give a positive answer to the question about the existence of a correlation among affective variables in the context of mathematics lessons aimed at preventing gambling, but we also recall that just a portion (10 %) of the interviewees use to gamble regularly: many of them had no to rare gambling experiences. This fact represents for us an element of strength of the research because findings do not apply only to (problematic) gamblers, but to a rather general population of secondary school students.

The context of gambling itself can be seen as either a limitation or a potentiality of the research. If we focus on its limiting character, we could say that it is possible that in other classroom settings, more tied to curricular mathematics, other kinds of relations among affective variables can emerge. However, we would like to say that the potentialities of this study reside in at least two of its features: (a) it focuses on an important social issue, about which schools can and want to play a part with respect to preventing gambling abuse, and about which mathematics teachers can play a significant role in teaching students how to understand betting games and their functioning (Andrà, Parolini & Verani, 2016); (b) it proposes a methodology that can be employed in order to investigate the relations among affective variables also in other contexts.

The methodology relies on multiple-choice and Likert-scale questionnaires, which have proved to be limited with respect to their ability to capture the complexity of affective issues and the psychological centrality of an individual's affective stances (Di Martino & Zan, 2011). We completely agree with Di Martino and Zan's (2011) standpoint, but in this study, we adopted such a method of data collection because we needed tools that allow us to observe inter-relations among variables at a large scale. We did so by resorting to calibrated and (proved to be) consistent instruments, developed by experts in the field. A follow-up, qualitative approach to the issues raised in this study can further confirm, provide nuances and even contrast the findings of our research.

The numerosity of our sample allows us to not only conclude that some affective variables are related, but also to prove that they are related in a certain direction: for example, that mathematics-related (wrong) beliefs increase as parental affective support decreases. Our study confirms that there is a relation between beliefs and behaviour, in particular that there is a strong positive correlation between misconceptions and gambling frequency, and the context of gambling represents a rare opportunity to investigate the relations between attitudes and beliefs as, up to our knowledge, it had not yet been investigated by researchers. Positive attitudes towards betting games increase as misconceptions increase (and as gambling frequency increases), but the higher the aversion with respect to betting games, the lower the misconceptions. Highest aversion has been observed among those students who declared to like gambling the most and the least, while intermediate positions with respect to positive attitudes correspond to lower aversion. It could be interesting to investigate these relations also with respect to mathematics and to other specific mathematical topics, but at the same time we are well aware, to this respect, of the peculiarity of the context of gambling (Andrà et al., 2016), and of the potential limitations in trying to extend these results to different teaching contexts.

Literature on mathematics-related emotions connect them to both beliefs and attitudes, but in this study, we investigate how (self-declared) emotional regulation is related to both the actions of gambling (i.e. its frequency) and beliefs. We found out that both emotional expressive suppression and cognitive reappraisal, namely regulation of emotions on the flow and before action, are positively correlated with beliefs, namely the more the misconceptions the more the regulation, but negatively correlated with gambling frequency, namely the more the gambling the less the regulation. As a cautious note to the methodology employed in our study, we add that Hannula (2012) warns us against the limitations of using statistical tools with strong assumptions on the linearity of relations that seem to have a much more complex nature. However, acknowledging this as a limitation of our study, we take the statistically significant correlations that emerge as an indicator that *also* these phenomena have features that can be captured by linear models.

## 6 Conclusions

We believe that our research, conducted at the boundary between Mathematics Education and Psychology, can have didactical consequences with respect to three main issues. Firstly, gambling and gambling abuse is spreading among young people all around the world and schools need to know not only the statistics about the phenomenon (e.g. how many 15-years-old pupils bet on a weekly basis), but also how their students' beliefs, attitudes and emotions, as well as the family context, contributes to either promote or discourage gambling practices so as to design and implement learning trajectories that truly help them develop awareness about the risks related to gambling abuse. Mathematics teaching plays a foreground role in this regard, as mathematical models allow one to understand the functioning of betting games, the probability of winning, their inequity and what happens in the long run.

Secondly, this study attempts to characterise the relations among different affective variables and to contribute to the understanding of mathematics-related affect as a system. It does so in a real-life setting, namely in a context where mathematical knowledge and misconceptions turn out to be significant for students' lives—not only for their mathematical understanding at school.

Thirdly, it does so by resorting to existing definitions of beliefs, conceptions, attitudes and emotions and by linking them to the methodological instruments chosen for the analysis. Results do not emerge from an experiment, but they are grounded in well-established theories of affect, which by themselves theorise existing relations between, for example, beliefs and behaviour, or emotions and beliefs, but fail to prove such claims with observations at a large scale. True, our study was exploratory in its nature. Follow-up ones are needed, so as to reveal further connections among affective variables and to allow us to know more about the systemic features of affect, which seems to be promising in allowing the researcher to grasp the interdependence among a variety of affective stances that emerge out of mathematics classrooms.

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# Relationship between affective-motivational constructs and heart rate

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The following survey study uses a quantitative research design to investigate motivational and affective aspects of students (aged 14–17) in a mathematical workshop on graph theory. Motivational and affective aspects are related to heart rate measurement (using the digital medium of a pulse watch) in mathematical knowledge development processes in an empirical-oriented mathematics class. Interestingly, a link between constructs on motivational and affective aspects and a heart rate measurement is describable. This gives further impulses for investigation and could be used in the future to determine the teaching phases or tasks in which students are particularly motivated.

Keywords: heart rate measurement, easiness, enjoyment, helpfulness, graph theory

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## 1 Introduction

Today, learning is understood as an active process (Scherer & Weigand, 2017). It is generally acknowledged that students constitute their mathematical knowledge in processes of action and negotiation (Krummheuer, 1984). In this context, students' mathematical knowledge development processes depend on individual domains of experience (Bauersfeld, 1988). With the approach of Burscheid and Struve (2020) these can be described as the construction of theories about certain phenomena of empiricism – empirical theories. With the cognitive psychology approach of "theory theory" by Alison Gopnik, it seems quite meaningful to describe the behavior and knowledge of students in theories. Burscheid and Struve (2020) state that with knowledge in this context not the knowledge formulated by the person is meant, but the knowledge that observers impute to the person in order to explain their behavior. The person – for instance students – behave as if they had the knowledge/theory (Burscheid & Struve, 2020). This can be related to Gopnik's (2003) studies when it states here that "children develop abstract, coherent, systems of entities and rules, particularly causal entities and rules, […] they develop theories" (Gopnik, 2003, p. 5).

With reference to these theorizations and knowledge development processes, this article presents a pilot study that aims to show the extent to which connections exist between affective and motivational aspects and the physiological component of a heart rate measurement in mathematical knowledge development processes in an empirical-oriented mathematics class. The importance of motivational and affective aspects – and not only exclusively cognitive factors – in teaching-learning processes first emerged in the mid-twentieth century (Goldin, Hannula, Heyd-Metzuyanim, Jansen, Kaasila, Lutovac, Di Martino, Morselli, Middleton, Pantziara & Zhang, 2016). In this context, Di Martino also states: "intentional actions involve complex relationships between affective and cognitive aspects; therefore, it is crucial to develop methods able to grasp this complexity" (Goldin et al., 2016, p. 3).

In order to clarify the question of the connection, the article first goes into the theoretical background with regard to the descriptive concept of domains of subjective experience (DSE for short) according to Bauersfeld (1983; 1985), empirical-oriented mathematics classes and how we want to deal with the term motivation in this article. With reference to current accounts regarding motivational and affective components in mathematics education, the section concerning *Research approaches and hypotheses* follows with a statement of the research purpose and clarification of the study's hypotheses. Methodological decisions regarding data collection and the setting used are made clear in Section *Methodology*, and the results and subsequent analysis are presented in Section *Analysis and results*. The article concludes with a review of our hypotheses, a summary of the results, and an outlook on the investigation of further relationships between instructional phases, tasks, and student motivation (Section *Discussion and conclusion*).

We would like to point out here that the results of this paper are also partly published in German in Pielsticker and Reifenrath (2022).

## 2 Theoretical background

Studies by Bauersfeld (1983; 1985), Coles (2015), Steinbring (2015), or Voigt (1984) could show that there are differences between teachers' perspectives and students' perspectives on mathematics lessons. Thus, Voigt states, the ambiguity of objects is not just a feature of individual episodes or individual tasks. The ambiguity of objects can be a fundamental and long-lasting feature of classroom talk when teacher and students interpret the objects in systematically different ways (Voigt, 1994). In this paper, we want to focus on the students' perspective and assume, in terms of a constructivist approach to learning, that students develop and negotiate their knowledge in dealing with the reality that surrounds them.

Decisive for this is the descriptive concept of domains of subjective experience (DSE) according to Bauersfeld (1988). In the article results and problems of microanalyses of mathematical classes, Bauersfeld (1985) presents his concept of DSE by formulating theses; simplified, these are individual cognitive and affective knowledge structures that students build up domain-specific. The starting point of a constructivist concept of learning and the approach of DSE according to Bauersfeld offers an orientation framework for our contribution.

With the concept of DSE we can describe how students develop their knowledge in a constructivist and interactionist sense. The core idea is that learning is a domain-specific process and thus can be described as bound to a particular situation and context. DSE include meaning, language, objects, and actions, which include cognitive and motivational or emotional dimensions. According to Bauersfeld, it can be stated that "[...] learning is characterized by the subjective reconstruction of social means and models through the negotiation of meaning in social interaction and in the course of related personal activities. New knowledge, then, is constituted and arises in the social interaction of members of a social group (culture), whose accomplishments reproduce as well as transmute the culture“ (Bauersfeld, 1988, p. 39). Language plays an important role in linking the individual DSE. According to Tiedemann (2016), a term has a meaning, especially in the context of other terms, but each term has a specific language use. In the sense of Bauersfeld's approach, affective knowledge structures play a decisive role in the development of mathematical knowledge in addition to the cognitive ones. Especially if we assume that a knowledge experience is "total" (Bauersfeld, 1985, p. 11). Rennie (1994) measures affective outcomes from visits to hands-on learning experiences in her study. Similar to Bauersfeld, she argues that in addition to cognitive outcomes, affective outcomes should also be considered for an overall picture of hands-on learning experiences. In her study, students visited a laboratory, which makes it a consideration of out-of-school learning opportunities. In this setting, Rennie sees motivation and willingness to engage as the most important affective outcomes. For example, the term motivation is used to describe the will-to-succeed across multiple contexts (Eccles, Wigfield, & Schiefele, 1998). A study by Renninger, which is addressing interest and motivation to learn in free-choice environments of informal science learning states: "Motivation to learn usually refers to the energy behind conscious decisions to achieve in school" (Renninger, 2007, p. 3). To describe affective outcomes like motivation, Rennie uses the components enjoyment, helpfulness, and easiness. We will follow the use of these components to describe the affective knowledge structures of the students in our study. In our context, this also involved hands-on learning experiences, which are based on the concept of empirical-oriented mathematics classes (Pielsticker, 2020). The workshop outlined in section *Methodology* was developed, designed and conducted in terms of empirical-oriented mathematics classes. The basis for empirical-oriented mathematics classes is the approach of empirical theories according to Burscheid and Struve (2020). Empirical-oriented mathematics classes is a teaching in which the teacher intendedly makes educational decisions to work with empirical objects as the mathematical objects of math classes. In mathematics classes, the empirical objects (e.g., 3D-printed tiles, graphs on a paper) do not intend to illustrate mathematical concepts that are abstract in nature, but they are rather the subject of the lesson (Pielsticker, 2020). The teaching-learning process from a mathematics class of this study takes this conception into account.

The purpose of this paper is to present, in a quantitative study, affective knowledge structures of students (through constructs measuring motivational and affective aspects), brought together with a heart rate measurement. Isoda and Nakagoshi (2000) had related students' emotional changes to the observation of heart rate in their study on problem posing in mathematics classrooms. In doing so, the authors note: Heart rate “is good quantitative indicator of the intensity of emotion” (Isoda & Nakagoshi, 2000, p. 89). Isoda and Nakagoshi looked at five types of changing emotional intensities in their study. For example, they observed a rapid intensity of changing heart rate when the students started to think in the given problem posing situation. Or a rapid and strong intensity of changing heart rate was observed when it came to social interaction (students exchange ideas and one gives an explanation). In our study we want to focus on motivational and affective aspects, combine them with a measurement of heart rates and describe affective knowledge structures.

## 2.1 Research approaches and hypotheses

Since affective and motivational aspects are important in view of our theoretical background from previous sections (and in doing so are especially related to the concept of DSE) it is useful to take a look at previous research and developed constructs in this area. A good overview of this is provided by Goldin et al. (2016), among others, who highlight that concepts and theories in the affective domain can be mapped along three dimensions. The first dimension is "identifying three broad categories of affect: motivation, emotions, and beliefs" (Goldin et al., 2016, p. 1). In this context, the last dimension comprises the theoretical level, which can be divided into three main levels: "physiological (embodied), psychological (individual), and social" (Goldin et al., 2016, p. 1). In this regard, Hannula also notes that affect in the mathematics context primarily addresses the psychological level, whereas the physiological level "is not very popular among mathematics educators" (Goldin et al., 2016, p. 2). In our study, we also address this level, among others, through heart rate measurement, thereby picking up on Hannula's observation that while some research exists in the area of affective and motivational components, there are also "insufficiently explored venues that call for additional research" (Goldin et al., 2016, p. 2). In this study, affective constructs are related to this same physiological component. In order to classify the affective constructs, we are investigating in this context, it is interesting to take a look at the widely used TMA model for the construct "attitude". According to this three-component model by Di Martino, attitude has three dimensions: "emotional dimension," "vision of mathematics," and "perceived competence" (Goldin et al., 2016, p. 6), which are interrelated. The constructs easiness, enjoyment, and helpfulness examined in our study, which will be explained in more detail below, can be located in the area of the emotional dimension and perceived competence. In addition, the constructs show a clear relation to motivation research. Goldin and others suggest, in addition to the general consideration of affective structures "that the focus of motivation research be shifted from the study of longer-term attitudes and beliefs toward that of in-the-moment-engagement" (Goldin et al., 2016, p. 18). Both with our collection of data on heart rate measurement and on easiness, enjoyment, and helpfulness, we take up the situational aspect (regarding "in-the-moment engagement") of motivation research. It is worth mentioning here that this distinction into "long-term" and "in-the-moment" is also found in many important individual motivational factors (interest, perceived instrumentality, etc.) (Goldin et al., 2016, p. 20). Thus, the concern of the study is to pick up motivational and affective aspects and to relate them to physiological components. Goldin and others also see the intersection of motivation and affect as "the future of the field" (Goldin et al., 2016, p. 23). In the present study, therefore, to measure motivational and affective aspects in knowledge development processes of students in empirical-oriented mathematics classes, we introduce the three constructs already mentioned, easiness, enjoyment, and helpfulness (Rennie, 1994; Woithe, 2020), and combine them with our measurement of heart rate. The choice of these constructs was largely based on a pilot study by Rennie (1994), which aimed to measure cognitive and affective outcomes in students attending a science education center and to develop a brief, easy-to-understand, and assessable measurement instrument for this purpose. The three constructs turned out to be central (Rennie, 1994). To capture the relevant constructs, we followed Woithe's items because, similar to our teaching-learning process, there was no explicit group work and therefore the items were adapted to the context (Woithe, 2020). In general, the survey context of Woithe, who studied the constructs in the context of workshops at the CERN S'Cool LAB, is also comparable to the teaching-learning process we developed in terms of the intervention duration of about 4.5 hours (Woithe, 2020) and via the structural design as a workshop. For the terms, at least for the constructs easiness and helpfulness, first a specification and for enjoyment a hint seems useful. According to Woithe (2020), the 2006 Pisa study rated "enjoyment of science" as “one of the measures of students' intrinsic motivation to learn science” (Woithe, 2020, p. 26). Here we can see our reference to the measurement of motivation. Helpfulness describes the perceived benefit of activities for students and is "closely related to value-oriented component of interest and serves as an indicator for a hold-component of interest development" (Woithe, 2020, p. 26). Regarding easiness, it can be said that it is related to "the cognitive load of the activities" (Woithe, 2020, p. 26) and Woithe states that “perceived easiness should not be too high to make sure activities cognitively activate students through challenging but doable tasks" (Woithe, 2020, p. 26). We therefore examined the following hypotheses in our study:

1. The heart rate (HR) deviates from the resting pulse of students in mathematical teaching-learning processes. (H1)
2. HR correlates with students' easiness in the given mathematical teaching-learning process. (H2)
3. Enjoyment correlates with easiness. (H3)
4. Helpfulness correlates with easiness. (H4)

In our first hypothesis H1, HR represents the abbreviation for heart rate or respectively the pulses in beats per minute (also used below). The hypothesis is based on a study by Isoda and Nakagoshi (2000), who demonstrated the importance of heart rate change in describing emotional change in students in a case study. Since we assume a change in emotional and affective components in the course of our mathematical teaching-learning process, this should also be shown by an increased heart rate. For our hypothesis H2, we look again at the study by Isoda and Nakagoshi. They state that “changing HR can be expressed in terms of arousal of the student’s mind” (Isoda &

Nakagoshi, 2000, p. 93) and that gradual increasing of HR represents concentrated thinking. The hypothesis picks up on the fact that this may also make the task perceived as easier. For our hypothesis H3, we can state: Woithe's study (2020) already analyzed a connection between helpfulness and enjoyment. As studies by Gläser-Zikuda and Mayring (2003) indicate, "if students enjoy learning [...] they are also more likely to perceive it as meaningful" (Woithe, 2020, p. 26). In hypothesis H3, we are interested in whether there is also a correlation between enjoyment and easiness. In H4, we wanted to know whether a correlation between helpfulness and easiness could also be established following the indications and results of the study by Woithe (2020) on helpfulness and enjoyment.

## 3 Methodology

### 3.1 Participants

Data were collected during several sessions of a workshop designed for this purpose in eight different learning groups. These learning groups were classes of secondary (German Sekundarschule) and high schools (German Gymnasium) in NRW Germany and represented grades 8 to 11, with the workshops taking place in their usual classroom environment. The workshop was designed to last 3,5 hours and was accompanied by two lecturers and the supervising teacher. Overall, the study thus has a sample size of N=73, with the analysis based on a sample size of N=46. The reason for the difference in the general sample size and the sample size used for the analysis is that the data (especially the questionnaires) were not complete (filled out) and usable for our analysis in every case.

### 3.2 Materials and measures

In order to be able to measure motivational and affective aspects in knowledge development processes of students in empirical-oriented mathematics classes, we methodically used a quantitative approach based on the measurement of the students' heart rate. A relationship between motivation and heart rate has already been demonstrated and researched in different studies. The data on this is initially not limited to the field of mathematics education but can be found primarily in sports science/sports medicine (Wang et al., 2015; Dadaczynski et al., 2017), as well as in computer science (Monkaresi et al., 2017), artificial intelligence (Patel et al., 2011), and psychology

(Scheibe & Fortenbacher, 2019). Specific to the field of mathematics education, studies on the relationship between heart rate and motivation can be found that take a case-study approach (Isoda & Nakagoshi, 2000), and on the relationship between perception of difficulty and motivation in arithmetic (Carroll et al., 1986). The latter is of particular interest to the present study because motivation is not defined as a concept that can be clearly grasped. Therefore, the study examined the three constructs easiness, enjoyment and helpfulness, which Rennie (1994) and Woithe (2020) have already used in similar workshops and research studies in advance. The heart rate was measured using the "Fitbit charge 4", a fitness watch which, as one of its functions, enables heart rate measurement over a definable interval and also outputs the data graphically.

### 3.3 Heart rate tracking devices – Fitbit

The pulse measurements and thus the activity and the demands on the students were carried out in this work with tracking devices. Fitbit Inc. is a US manufacturer of electronic vital function measuring devices, so-called "tracking devices", founded in May 2007 in San Francisco, California (https://www.fitbit.com/global/us/home). Tracking devices of the first generations still showed a huge variance in measurement accuracy whereas newer generations of tracking devices show a deviation from tests under laboratory conditions of only up to 4 % (University Aberystwyth, 2019). Numerous studies now prove that tracking devices have a positive effect on the likelihood of success in implementing a healthier lifestyle (Ridgers, McNarry, & Mackintosh, 2016). Each of our trackers had online access to the Fitbit website. In addition, all data was anonymized and worked with, for example, Tracker A, Tracker B, etc. The data such as heart rate, date and time could be retrieved via the website and the respective online access of the tracker at Fitbit. Accordingly, this data could be exported to Excel and used for the analysis. The representation in Figure 1 is based on the graph from Fitbit.

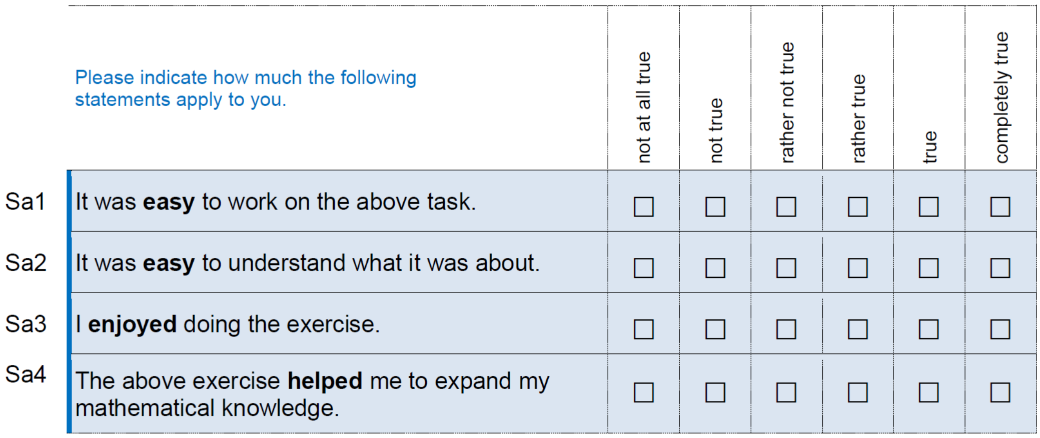

Figure 1. Data and representation of the values of the trackers
(based on https://www.fitbit.com/global/us/home).

### 3.4 Resting pulse

Another important reference for our analysis is the resting pulse. The resting pulse is the pulse at which no activities or movements are performed. This means that it is the pulse measured in the absence of physical and mental stress. Ideally, the resting pulse is measured immediately after waking up. The resting pulse rate of adult's averages around 70 BPM, while that of adolescents averages around 80 BPM (Pape, Kurtz & Silbernagl, 2005). Since the analysis of this paper was conducted with adolescents in the eighth and ninth grade, the resting pulse rate of 60 to 100 BPM is decisive (Healthwise Staff, 2020). The present pulse values are therefore not a resting pulse in the true sense of the word. Nevertheless, a pulse can be read that is very close to a resting pulse. In the processing phases, the students reach phases in which they have a relaxation pulse (see Figure 3). This means that the pulse flattens out and becomes calmer.

### 3.5 Affective-motivational constructs

To measure motivational and affective aspects in knowledge development processes of students in empirical-oriented mathematics classes, we bring together the three constructs and our measurement of heart rate. The separate constructs easiness, enjoyment and helpfulness were recorded using a sequenced questionnaire integrated into a workbook. In total, this consisted of 19 items that could be assigned to the constructs in groups and related to different phases of the workshop. The constructs were always queried after the respective phase to obtain as fresh an impression as possible. In addition, each of the 19 items has an even number of characteristic values (1 "not at all true" - 6 "completely true") in order to avoid an "error of central tendency" with Likert scales. An example of the first four items can be found in Figure 2. (Sa1: It was easy to work on the above task, Sa2: It was easy to understand, what it was about, Sa3: I enjoyed doing the exercise, Sa4: The above exercise helped me to expand my mathematical knowledge).



| | Please indicate how much the following statements apply to you. | not at all true | not true | rather not true | rather true | true | completely true |
|---|---|---|---|---|---|---|---|
| Sa1 | It was **easy** to work on the above task. | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |
| Sa2 | It was **easy** to understand what it was about. | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |
| Sa3 | I **enjoyed** doing the exercise. | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |
| Sa4 | The above exercise **helped** me to expand my mathematical knowledge. | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |

Figure 2. First four items of the sequenced questionnaire (translated from German to English).

The reliability of the scales was satisfactory; the corresponding *Cronbach's alpha* is shown in Table 1. According to Field (2017) and Hair et al. (2018), the values for *Cronbach's alpha* are in the good range (> 0.7) and would not improve by removing an item. In this sense, the items on the constructs (also according to Rennie, 1994; Woithe, 2020) measure what they are supposed to measure and are consistent. The construct easiness had a higher number of items because of its importance for our hypotheses.

**Table 1.** Reliability of the scales.

| Constructs | Number of items | Cronbach's alpha |
|---|---|---|
| Easiness | 9 | .782 |
| Enjoyment | 5 | .789 |
| Helpfulness | 5 | .728 |

Both data series were quantitatively recorded in different learning groups. For preparation and analysis of the data set we used SPSS. The assignment of the data series was done via an anonymized code.

## 3.6 Procedures

Data collection took place between August and December 2020. In terms of subject matter, the workshop is based on graph theory and addresses central questions about the functioning of a navigation system and the determination of optimal paths. The subject area was chosen, among other things, because it requires little prior experience on the part of the learning group and is easy to elementarize to a certain extent.

The workshop's content was divided into three sections, which can also be found in the workbook that the learning groups received. The first part deals with basic concepts of graph theory and clarifies them, whereas the second part focuses on the optimization of graphs and paths and leads to the third part, which deals with the application of algorithms. The workbook not only fulfilled content-related purposes but also served data collection purposes, since in addition to the content, which was prepared in appropriately sequenced boxes (information boxes, elaboration boxes, exercise boxes), the sequenced questionnaire items for recording the constructs easiness, enjoyment and helpfulness could also be found after the respective phase. The measurement of the pulse data within the workshop or the time of their collection followed a defined pattern, which fits into its organizational conception (see Figure 3). In the entire teaching-learning process, a short introduction for the learning group to the functioning of the heart rate monitor took place in the introductory phase. The start of the heart rate measurement was then after the creation of the workbook code before the start of the content work and the first part. During the following sequence, heart rates were constantly measured until the end of the workshop or the teaching-learning process. An exception was the breaks, in which the heart rate measurement could also be paused. A decisive factor was the resting pulse's determination as a comparative value for the heart rate and its changes. This took place after the end of the workshop and before the heart rate monitor was switched off, as the drop in heart rate was clearly visible in the data.

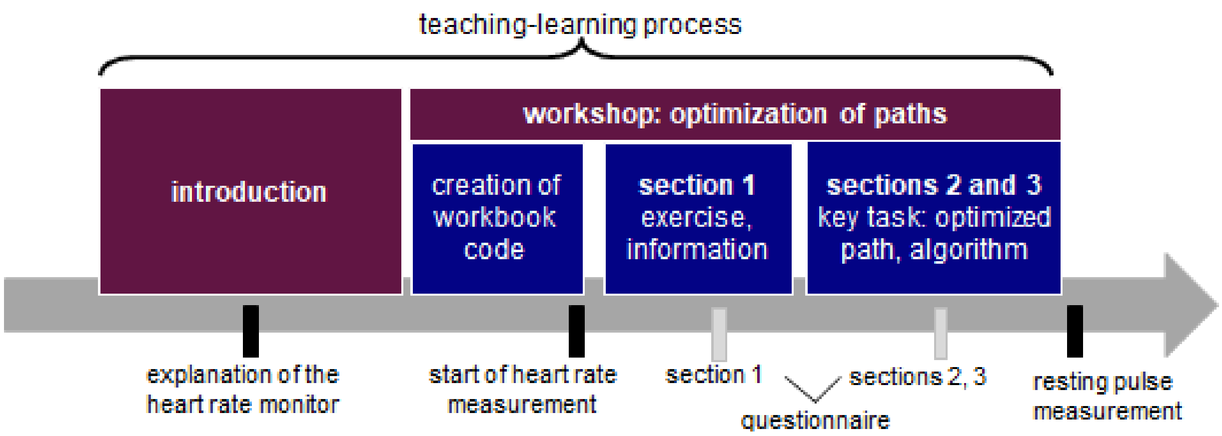



Figure 3. Organizational structure of the teaching-learning process.

# 4 Analysis and results

## 4.1 Descriptive results and correlations

In preparation for our *T-test*, we checked our data set on HR for metrics and normal distribution (Field, 2017; Hair et al., 2018). We can state that the heart rates are metric and normally distributed. The entire sample (N=46) was tested using the Shapiro-Wilk test (the value was > 0.1). Moreover, with Table 2, we can confirm the *first hypothesis* (*H1*), “The HR deviates from the resting pulse of students in mathematical teaching-learning processes” for our data set. For the students, the HR in our mathematical teaching-learning process deviates from the resting pulse of the latter. The significance is <0.01. For illustration purposes, a boxplot of the pulse data of a single participant with a resting pulse of 79 bpm is shown in Figure 4 for the entire teaching-learning process. The red line corresponds to the resting pulse and is well below the median for the specific case, for example.

**Table 2.** Data of the T-Test.

| Variable | Average | N | p value |
|---|---|---|---|
| Resting pulse | 70.33 | 46 | |
| Pulse (entire workshop) | 87.89 | 46 | |
| Difference | 17.6 | | <.000 |

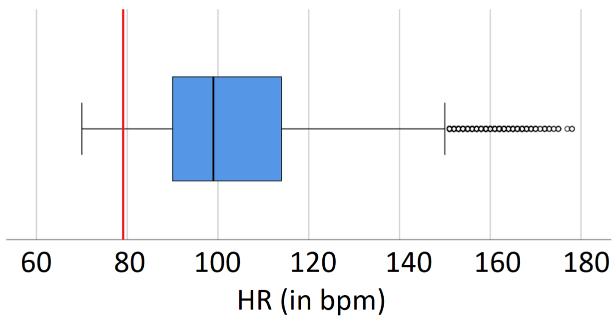



Figure 4. Boxplot of the pulse data of a single participant with a resting pulse of 79 bpm (indicated in red) for the entire teaching-learning process.

Furthermore, we examined five variables for our analysis (see Table 3). The value of the HR for the entire workshop (also known from the *T-test*), easiness, which is relatively high with a mean value of 5.07 on a straight scale from 1 ("not at all true") to 6 ("completely true"), enjoyment, and helpfulness (with a mean value slightly below easiness) and gender. Regarding the latter, we can state that more male test persons participated. Here, the gender "female" was coded by 1 and the gender "male" by 2. The gender "diverse" was not specified by the participants.

**Table 3.** Descriptive data for evaluation of the workshop.

| Variable | Average | N |
|---|---|---|
| Pulse (entire workshop) | 87.89 | 46 |
| Easiness (entire workshop) | 5.07 | 46 |
| Enjoyment (entire workshop) | 4.79 | 46 |
| Helpfulness (entire workshop) | 4.17 | 46 |
| Gender | 1.57 | 46 |

We then calculated a *Pearson correlation* to check whether there is a correlation between the variables. This was done mainly in preparation for a subsequent regression analysis. Correlations significant at $p \leq 0.05$ are indicated in bold and *Pearson correlation* coefficients are used for correlations between metric variables. According to the effect sizes according to Cohen (1988), we can then state in our analysis (correlation matrix, Table 4): Easiness and enjoyment correlate with a medium effect (above 0.5), as do helpfulness and enjoyment. Furthermore, helpfulness and easiness correlate with a small effect (up to 0.5).

**Table 4.** Correlation matrix.

| | Variable | N | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| 1 | Pulse (entire workshop) | 46 | 1 | | | | |
| 2 | Easiness (entire workshop) | 46 | .247 | 1 | | | |
| 3 | Enjoyment (entire workshop) | 46 | .024 | .560 | 1 | | |
| 4 | Helpfulness (entire workshop) | 46 | -.133 | .352 | .574 | 1 | |
| 5 | Gender | 46 | -.084 | -.100 | -.075 | -.247 | 1 |

Below in Figure 5 is a linear regression to our value 0.560 (see Table 4).

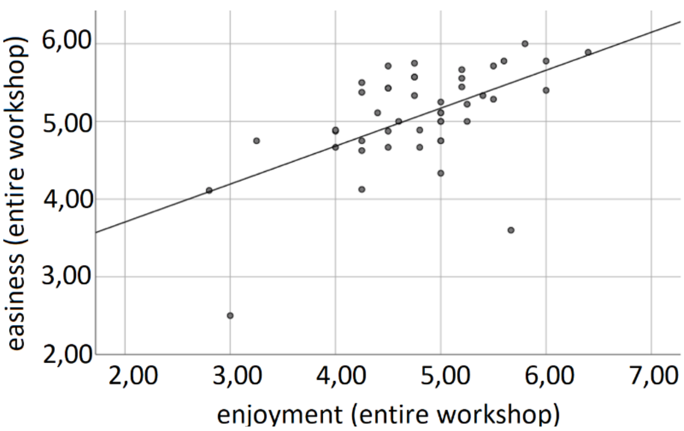


Figure 5. Simple scatter plot with linear regression of easiness and enjoyment (graphic belongs to value 0.560).

In our multiple regression analysis (see Table 5), we address *H3* "Enjoyment correlates with easiness" and *H4* "Helpfulness correlates with easiness" in addition to *H2* "HR correlates with students' easiness in the given mathematical teaching-learning process".

**Table 5.** Multiple regression analysis on the dependent variable easiness (entire workshop). Here: * p < 0.10; ** p < 0.05 (see Hair et al., 2018 for significance level α).

| Independent variable | Standardized coefficients $\beta$ | p value |
|---|---|---|
| Pulse (entire workshop) | .246 | .059* |
| Enjoyment (entire workshop) | .499 | .002** |
| Helpfulness (entire workshop) | .093 | .560 |
| Gender | -.019 | .884 |
| $R^2$ | .375 | |
| Corrected $R^2$ | .314 | |
| N | 46 | |

We can confirm our *second hypothesis* (see Table 5). The independent variable HR correlates with the dependent variable easiness. We see a positive correlation, with a beta value of 0.246 and a significance of $p<0.1$. We can therefore say that a higher HR correlates with a higher perception of easiness. According to the work of Khamis and Kepler (2010) and Hair et al. (2018), we can state with $N=5k+20$ that we can include four independent variables for the regression calculation (for a "statistical power"). Our four variables are HR, enjoyment, helpfulness and gender. This brings us to the confirmation of *H3*. With a positive beta value of 0.499 and a significance of $p<0.05$ (see Table 5), we can conclude that: A higher the enjoyment correlates with a higher perception of easiness. Unfortunately, we have to reject our *hypothesis H4* and cannot confirm it in our regression. There was no significance here with $p=0.560$. Furthermore, it should be mentioned that gender serves as a control variable to eliminate this effect from our regression and also to control for it.

## 5 Discussion and conclusion

For our four hypotheses regarding the measurement of motivational and affective aspects in knowledge development processes of students in empirical-oriented mathematics classes, we can conclude as follows: We were able to confirm our *hypotheses*

*H1, H2* and *H3*. The confirmation of *H1* might be seen as an indication for cognitive engagement with regard to Isoda's and Nakagoshi's statement that "changing HR can be expressed in terms of arousal of the student's mind" (Isoda & Nakagoshi, 2000, p. 93) and that gradual increasing of HR represents concentrated thinking. The confirmed correlation in *H2* between heart rate and perceived easiness can be interpreted as follows. Presumably, hormones such as adrenaline, noradrenaline or cortisol, for example, lead to an increase in performance, but this was obviously perceived as positive. This is so-called eustress. This is positive stress because, for example, the tasks were performed well, a result was obtained, or one feels good and confident in the situation. We had to reject *H4* for our data set. We can confirm the results of Rennie (1994) and Woithe (2020) and extend them by measuring the constructs in a mathematical workshop. We have thus placed the three constructs in a different context and expanded them. We have also examined the results of Isoda's and Nakagoshi's (2000) case study in the area for N=46. A higher number of subjects would certainly be useful for more detailed statements. Naturally, our results are subject to some limitations. First, we used our three constructs and the heart rate to infer learner motivation. There were only a limited number of items for each construct in the survey study. For further research it would have to be considered that not only the constructs easiness, enjoyment and helpfulness but also many more aspects need to be considered when investigating motivational and affective aspects. Second, the heart rate can also be dependent on other factors. For instance, variables like time of measurement, after or before a meal, previous school lesson (e.g., physical education), breathing, age, body size or also grades of the students. The age of the students would be another interesting variable. Third, our analyses are based on a relatively small sample of 46 students. Despite these limitations, we believe that our results are valuable to discussion in mathematics education. It is one of the first quantitative studies to bring together constructs for measuring motivational and affective aspects (in an empirical-oriented mathematics classes) with a heart rate measurement (and thus digital tools); thus providing the linkage of affective constructs and physiological components addressed in Section *Research approaches and hypotheses*, addressing Hannula's "insufficiently explored venues that call for additional research" (Goldin et al., 2016, p. 2). In this regard, our results show that they provide an extension for already established constructs describing motivational and affective aspects by heart-rate measurement and put them on a broader basis for discussion.

With this and beyond, the study offers future links. For example, it is possible to add facial features similar to the case study by Isoda and Nakagoshi (2000) or to combine the results on affective knowledge structures with a cognitive dimension in the concept of DSE according to Bauersfeld (1988). In the long run, it would be interesting for (mathematics) teachers to know which phases of the lesson or which tasks (e.g., problem solving or drill training) particularly motivate learners. Thus, we hope that our results provide some valuable insights for further studies of motivational and affective aspects in mathematics education.

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# Attitudes in mathematical discovery processes: The case of Alex and Milo

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This paper's purpose is to investigate the attitude of students in mathematical discovery processes in terms of the handling of counterexamples. By understanding this attitude as a kind of scientific attitude, it consists of different aspects that become visible in the behaviour during a mathematical discovery process. Since such a process is particularly complex, the author's interest is to use the concept of attitude as an explanation for students' behaviour that occurs when dealing with conflicts such as counterexamples. Semi-structured interviews with sixth graders of a German Gymnasium were conducted and analysed in a qualitative and interpretative way. As a result, the case study of Alex and Milo is presented. Based on the framework that observable behaviour is influenced by an underlying attitude, there are drawn conclusions about Alex's and Milo's attitudes adopted in the mathematical discovery process and their impact on the process is elaborated.

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## 1 Theoretical framework

### 1.1 Mathematical discovery process

> "The learning of mathematics is more effective [...] the more it is done in the sense of one's own active experiences [...]."

Winter's (2016, p. 1) quote is based on a constructivist view of learning, namely that learners are supposed to take an active role in the learning process while the teacher provides a suitable learning environment (Kunter et al., 2013). In terms of the teaching and learning of mathematics, this goes along with Freudenthal's (1973) idea of the so-called guided reinvention, so that learners experience mathematics as an activity rather than a ready-made product. In this way, learners are supposed to take an active role and experience the process of discovering and developing mathematics rather than being confronted with just its results. These processes of discovering and developing mathematics is what this paper refers to as *mathematical discovery processes*. Thus, the term *mathematical discovery processes* involves activities that are usually performed by research mathematicians. So, in this way, it is not only about discovering but also about questioning and reasoning to gain new knowledge.

This paper refers to the model for mathematical discovery as quasi-empirical experimentation, which is seeing mathematics as kind of experimental science that deals with abstract objects such as numbers or relations (Leuders & Philipp, 2013). In that sense, by zooming in on the process of mathematical discovery, *mathematical discovery processes* can consist of the following activities: *generating examples*, *structuring* based on relevant characteristics, *developing hypotheses* and *testing and proving* them (based on Philipp, 2013). Within these activities, there are numerous barriers to overcome in order to gain new knowledge and each of those sub-processes can be a great challenge for learners as different study results underline (e.g., Dunbar & Klahr, 1989; Kuhn et al., 1988; Kuhn, 1989):

In mathematical discovery processes, learners tend to propose a hypothesis after only one example. Moreover, they often conduct one single experiment to be convinced that their hypothesis is correct. In contrast to that, learners have difficulties in deciding what evidence is sufficient to reject their hypothesis. At the same time, learners tend to ignore evidence that is inconsistent with their hypothesis or try to gain some evidence that would confirm it. In general, students seem to test their hypothesis in order to find confirming evidence instead of checking the correctness. Tweney (1989) even revealed a general strategy in dealing with hypotheses: people tend to generate evidence that confirms the hypothesis first. Once there is enough evidence gained, people try to look for counterexamples or attempt to disconfirm the hypothesis.

This paper focuses on conflict situations that are most likely to arise during a mathematical discovery process. The way of dealing with those situations is crucial for gaining knowledge. According to Bauersfeld (1985), a conflict is a situation that does not fit into the learner's cognitive frame or "subjective domains of experience" (p. 11) as they refer to it. Therefore, a conflict is a situation, for instance a counterexample, that is not compatible with the existing hypothesis. As mentioned before, some learners tend to ignore evidence that is inconsistent with their hypothesis. Besides of that, studies have shown that counterexamples or contradictions in general also led to a reinterpretation of the evidence and not to a modification of their hypothesis (Kuhn, 1989). Furthermore, when counterexamples were really perceived as counterexamples, they were not considered to be sufficient for disproving a hypothesis (Kuhn et al., 1988). This behaviour seems to be worth analysing in detail with regard to the underlying attitude that learners take in mathematical processes to eventually gain a deeper understanding of its impact on doing mathematics. In order to meet this concern, we will first take a closer look at the concept of attitude in general and the way it is used in this paper.

## 1.2 Attitudes

As many authors have already stated, there is no universal definition of the term or concept of attitude (e.g., Pepin & Roesken-Winter, 2015; Walsh, 1991). While some earlier studies referred to attitude as a general concept overarching all mathematical topics and activities (e.g., Haladyna et al., 1983), it seems to be common ground nowadays that attitude depends on the objects and situations an individual is faced with (Kulm, 1980). Moreover, attitude can not only be regarded as a single dimensional construct but rather multi-dimensional comprising cognitive, affective, and conative or behavioural aspects (e.g., Di Martino & Zan, 2010). This gave rise to the idea of a "working definition" (Daskalogianni & Simpson, 2000, p. 217), so that the concept of attitude depends on research interest and situations to be studied. With regard to this proposal, I first take a brief look at attitude in mathematics education literature before I then derive an understanding of the concept of attitude suitable for this paper's interest.

In his pioneering work concerning affect in mathematics education, McLeod (1992) described attitude, in addition to beliefs and emotions, as a key affective construct. Later, Goldin (2002) added values, ethics, and morals as a fourth component. When considering stability and intensity, both researchers classified attitudes somewhere in between beliefs and emotions. In this context, beliefs as the most stable and emotions as the most intense of the three constructs form the two poles, between which attitudes can be classified as "feelings of moderate intensity and reasonable stability" (McLeod, 1992, p. 581). On the one hand, one's attitude towards an object or a situation seems, therefore, to be a moderately stable construct but, one the other hand, still has the potential to be modified (Liljedahl et al., 2010).

In line with the perspective of social psychology, attitude can be seen as a trait of an individual that influences their behaviour (Allport, 1935). Since attitudes are moderately stable, they manifest in "manners of acting, feeling, or thinking" (Philipp, 2007, p. 259). By considering attitudes as a concept that one's behaviour is based on, "they may involve positive or negative feelings" as Philipp (2007, p. 259) stated, but seem to be more than an evaluative judgement about an object or in this case, a disposition towards mathematics.

As previously mentioned, *mathematical discovery processes* can be seen as a way of conducting an experiment. That view highlights the dynamic character of mathematics as an evolving science like natural sciences. It is therefore worthwhile to look at the concept of attitude from this point of view as well. In the field of science education, Gardner (1975) proposed a fundamental distinction that is also suitable and probably even necessary for the field of mathematics education. He distinguishes the terms "attitude to(wards) sth." (p. 1) and "adjective + attitude" (p. 1). In his case, the adjective in the second term can be replaced with scientific, while in mathematics education we might call it *mathematical attitude*. The first term always includes some attitude object to which the respondent is invited to react favorably or unfavorably, for instance attitude towards mathematics or attitude towards problem solving. The second term is understood as ways or styles of thinking, acting or behaving, which influence the way we behave in certain situations and it's the meaning which this paper is based on. In this way, attitude has an influence on behavior and the other way around, conclusions can be drawn about attitude from behavior.

In the field of science education, great efforts have been made to characterise a desirable scientific attitude due to its importance for supporting scientific learning and enhancing the performance of students' scientific activity. According to the American Association for the Advancement of Science (1993) scientific attitude generally includes: *curiosity*, *honesty*, *open-mindedness* and *doubt*. Other researchers add further characteristics such as *respect for data*, *diligence*, *creativity*, *cooperation*, and *confidence* (Harlen, 1996; Anderson, 1980). Transferring these considerations to the field of an idealised attitude in mathematical discovery processes, it becomes clear that attitude in this case is a multi-dimensional construct. In this case, the term *scientific attitude* is used in a normative way, so that it is understood in the sense of a desirable attitude. In the study presented in this paper, the term *attitude* will be used in a descriptive way in order to characterize attitudes that students actually adopt in mathematical discovery processes. Thus, students' behaviour in a mathematical discovery process is seen in this paper as the outward expression of an attitude, so that attitude itself is not a directly measurable construct. However, it is possible to draw conclusions about underlying attitudes based on observable behaviour.

### 1.3 Research questions

As pointed out before, this paper assumes that a learner's observable behaviour in a mathematical discovery process is based on the attitude the learner adopts during the process. In order to gain a deeper understanding of learners' mathematical discovery processes, one aim of the study this paper is based on is to draw conclusions about those different attitudes. As space is limited, this paper especially focusses on a typical situation that might arise in the course of a discovery process: the emergence of counterexamples, contradictions or objections and how learners deal with it. Therefore, this paper addresses the following specific research questions:

1. What is the behaviour of the two students Alex and Milo when dealing with conflicts (such as counterexamples) in the shown excerpt of the interview?
2. To what extent can conclusions be drawn from the behaviour about the students' attitudes in dealing with counterexamples during a mathematical discovery process?

## 2 Method

The data was collected in an exploratory semi-structured interview with twelve sixth graders of secondary school (German Gymnasium). This paper focusses on the case study of the two students Alex and Milo, who were interviewed together. Their interview took place in October 2020 and was conducted by the author. The interview took about 70 minutes and was designed to simulate a mathematical discovery process with low level of interviewer intervention. For gaining an insight into the students' thinking process, the *think aloud method* was used (Ericsson & Simon, 1993). The students worked in tandem in an interactive situation on an explorative task about sums of successive natural numbers adopted from Leuders et al. (2011). To be more precise, the students' task was to develop a 'trick' how to easily decide whether a given number is a so called *staircase number* (a number, that can be represented as a sum of successive natural numbers). The task requires basic mathematical knowledge but, at the same time, it offers a lot of opportunities for making discoveries, conjecturing and reasoning. For instance, students could assume that **all** numbers are staircase numbers, all **odd** numbers are staircase numbers, all **even** numbers are staircase numbers, **not all** even numbers are staircase numbers or that **all** numbers are staircase numbers, except 2, 4, 8, 16, 32, ... and so on.

From a mathematical perspective, the characteristic this task is looking for is a number (not) being a power of two. So, numbers that are power of two are **not** staircase numbers, all the other numbers are staircase numbers. With that in mind, some of the previously presented hypotheses are wrong or at least need to be modified. Of course, the task does not want the students to use the term *power of two*, since it has not yet been part of their mathematic class so far. However, this characteristic can be discovered and justified, for instance, by using the small round plates (see Figure 1). Nevertheless, it was not intended for the students to solve the task completely but to evoke the aforementioned mathematical processes, so that activities like generating and exploring examples, structuring, developing hypotheses as well as testing and proving them can take place.

At the beginning of the interview, the term *staircase number* was clarified by using enactive representations with small round plates, iconic representations with a dot pattern on squared paper and arithmetic representations of the number 25 (see Figure 1). The students could optionally use all of them during their discovery process.

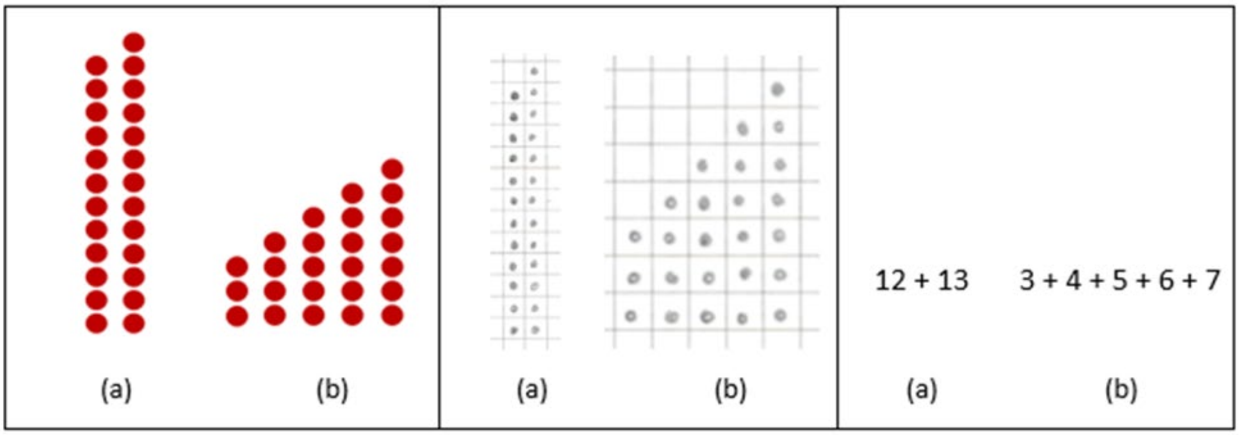


Figure 1. Enactive (with small round plates), iconic (dot pattern) and arithmetic representation of the number 25 as a (a) two-step and (b) multi-step 'staircase number' (own representation).

The interviews were videotaped and transcribed. Following a qualitative research approach the aim of the study is to draw conclusions about students' attitudes adopted in a mathematical discovery process. For analysing the data, a structuring qualitative coding method was initially used to categorize the behaviour in dealing with conflicts to get an overview of the different kinds of students' reactions (Mayring, 2015). Categories have been gained both deductively on the basis of the theoretical background and inductively to further differentiate them in terms of research interest (see Table 1 for an excerpt of the category system). The coding was carried out twice by the author. In order to draw conclusions about the attitude of the students from their behaviour, crucial scenes were analysed by a turn-by-turn analysis following an interpretative research paradigm (Voigt, 1984). The aim of this approach is to generate hypotheses that explain phenomena in teaching and learning mathematics in the sense of *abduction*. This means that, starting from a phenomenon, a general rule is set up that together with the recognition of the case at hand causes the phenomenon (Peirce, 1958, as cited in Meyer, 2018). The overall aim is to "make sense" (Eisenhart, 1988, p. 103) in accordance with the method of objective hermeneutics by making cognitive processes visible. The aim of this approach is to generate hypotheses that can be further investigated in future research.

To answer the research questions, an analysis with particular focus on each learner was first carried out and then the interaction and joint mathematical process were considered. For the sake of clarity, this paper only presents the results that have proven to be plausible within the analysis (Krummheuer & Brand, 2001, p. 90).

**Table 1.** Category system as a result of the qualitative content analysis

| Category | Anchor example | Coding rules |
|---|---|---|
| **Review of conflict trigger** | Milo: so first of all, here's one. that's two that's three. (points at first two steps of 1\|2\|3) | The conflict trigger (e.g., counterexample) is checked. |
| **Rejection of hypothesis** | Milo: I think this one is right. (points at the first hypothesis) there must always be three or more small plates- but not this. (points at the second hypothesis) | The hypothesis is completely rejected and is not pursued further in a modified form (otherwise: modification of hypothesis) |
| **Modification of hypothesis** | see subcategories | Also includes a rejection of the original hypothesis (in this way the hypothesis is false), but the hypothesis is pursued in a modified way. |
| ***Classification*** | Alex: so there are different forms of staircase numbers. namely this one (points at 1\|2\|3) and then this one. (lays 1\|2) | A classification takes place with regard to a characteristic, which specifies the hypothesis. |
| ***Exclusion of cases*** | Alex: [...] twelve is an exception. | The cases that contradict the original hypothesis are excluded or named as exceptions. |
| **...** | ... | ... |
| **Cancellation** | Alex: how difficult is that? [...] eh? i don't understand it anymore. | No specific rejection of the hypothesis, but termination of the entire process. |
| **...** | ... | ... |

# 3 Results

In the following I will take a closer look at the case study of Alex and Milo (names are pseudonyms). An excerpt of the interview with Alex and Milo and its corresponding interpretation is presented. In the transcript, the coding is added as well. In the same way as the analysis was carried out, here the individual students Alex and Milo are considered first, before a brief comment is made on the joint mathematical discovery process. At this point, it is important to note that the analysis of one single scene of the interview does not give enough information about the students' attitudes. For this, the behaviour of the students during the entire process must be included to draw conclusions about an underlying attitude. However, this scene and its interpretation can at least give an impression of it.

Note on notations in transcript: the expression *lays 1|2|3* is the written representation of the act to lay the small plates as a staircase with three steps of height one, two and three small plates.

## 3.1 Excerpt from the interview

Here, Alex and Milo have formulated and noted two hypotheses: *(1) There must always be three or more small plates* and *(2) Only odd numbers can be formed into a staircase*. Immediately before the excerpt begins, Milo has placed the arrangement 1|2|3 with small plates. Then the following scene takes place (see Figure 2). The code *conflict arises* is not a code of the category system but to make the conflict situation clear to the reader.

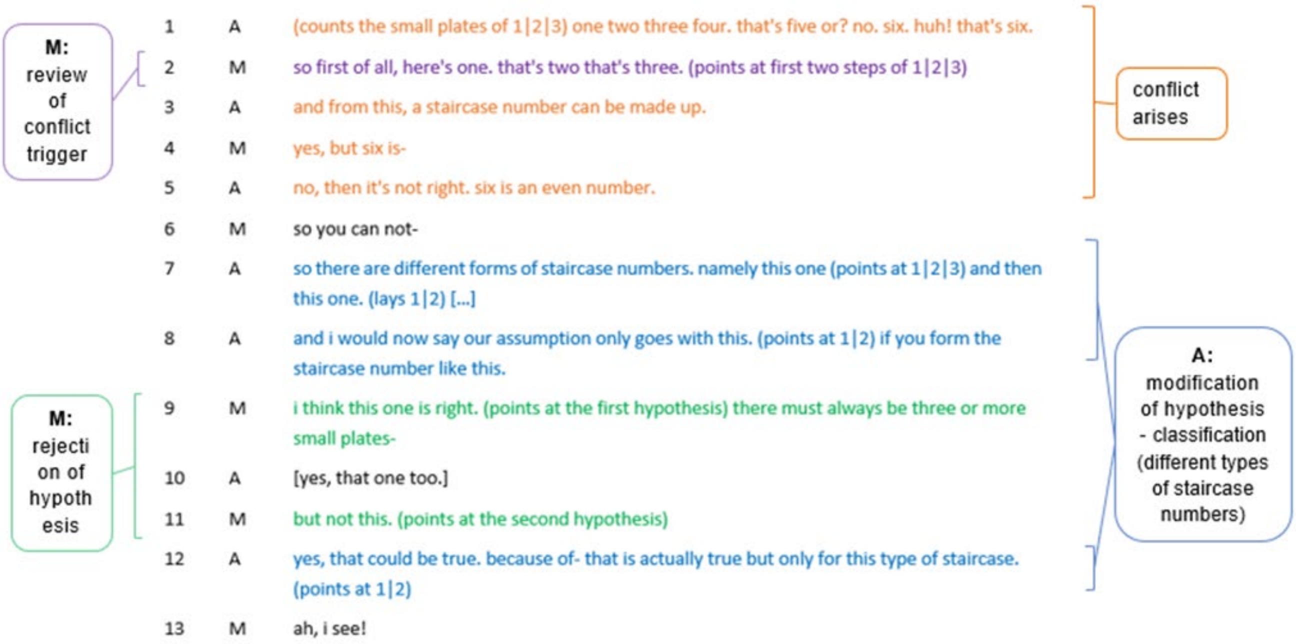


Figure 2. Excerpt from the interview with Alex and Milo with coding (own representation).

## 3.2 The case of Alex

In turn 1, Alex seems to be surprised when he recognises that the staircase Milo constructed adds up to six. He is convinced of the hypothesis that only odd numbers can be staircase numbers, so that the counterexample six does not fit into his theory. Nevertheless, in turn 3, he states the counterexample to be correct so he perceives six as a counterexample. On that basis, he puts the counterexample in relation to their hypothesis and states the hypothesis to be incorrect ("then it's not right", turn 5). He justifies the disconfirmation with the parity characteristic of six. It is striking, that at this point for Alex the occurrence of a counterexample is the trigger to make a *classification of different types of staircase numbers*. He distinguishes between two-step staircases (the type of staircases that has occurred up to the present scene) and the type of staircases to which he assigns the counterexample six. In this situation it is not totally clear which type of staircases he refers to by the latter: it could be multi-stage staircases starting with the height of one plate as well as multi-stage staircases in a more general way. It could also be the case that Alex himself is not quite clear about it.

In the following, Alex's classification is the starting point for a specification of the hypothesis *only odd number can be formed into a staircase*. Although he has previously falsified the hypothesis (turn 5), he still maintains and even further develops it by specifying the class of types of staircases to which the hypothesis refers (turn 8). Thus, for Alex, the hypothesis *only odd number can be formed into a staircase* turns into *only odd number can be formed into a two-step staircase*. The specified form of the hypothesis shows Alex's way of resolving the conflict created by the counterexample. His conviction of this approach is shown in the fact that he defends it against Milo's objection (turn 12).

Alex's behaviour in this excerpt shows some characteristics that indicate a more general attitude he adopts in mathematical discovery processes. He shows great conviction with regard to the hypothesis that has been made. In the course of the scene it also becomes clear that Alex literally sticks to it. When he recognises that the counterexample contradicts the hypothesis, he does not reject it but accepts the counterexample and modifies the hypothesis to integrate it. It is remarkable that Alex uses the typical mathematical activity of classification for this purpose. In summary, Alex's attitude in this excerpt can be described as persistent, which is also confirmed in the further course of the interview. A counterexample does not make him abandon the hypothesis but rather taking it as a trigger to develop the hypothesis further. For this attitude, counterexamples have a great potential for mathematical discovery processes.

### 3.3 The case of Milo

Milo is the one who has placed the arrangement 1|2|3 with the small plates. When Alex detects it as a counterexample to their hypothesis *only odd number can be formed into a staircase*, Milo's first reaction is to *check the counterexample by* accurately recounting the small plates of 1|2|3 (turn 2). In the following (turn 4 and 6), he does not really get a chance to verbalise all his thoughts, but due to his further behaviour one can assume that he accepts the counterexample as such, just like Alex does. What is striking is that Milo's handling of the counterexample differs from Alex's. Milo refers directly to the two hypotheses they had previously made. By reinterpreting the counterexample as a confirmation example for the first hypothesis (*There must always be three or more small plates.*), he approves it. In contrast to that, six as a counterexample is the decisive point for *rejecting the second hypothesis* (*only odd number can be formed into a staircase*). He does not make an attempt to resolve the conflict other than strictly disconfirming the hypothesis. Because of the counterexample, the hypothesis has come to an end for Milo at this point. This is particularly clear in the way he contrasts the two hypotheses: in turn 9 he starts his sentence by confirming the first and then clearly ends in turn 11 with the statement that differentiates the second hypothesis as incorrect. The possibility of a further development does not seem to be given until Alex suggests it. The surprise in Milo's statement confirms this interpretation (turn 13). Although for him this solution was not an option as a way out of the conflict, he accepts Alex's proposal and supports the specification in the following.

Like Alex, Milo's behaviour also indicates a certain attitude he adopts in the course of the mathematical discovery process. Although Alex and Milo have set up the two hypotheses together in advance, Milo does not show the same persistent behaviour that Alex does. On the contrary, Milo shows a sceptical attitude towards the hypothesis that is made clear in the significance of the hypothesis for him. As soon as a counterexample occurs, the hypothesis is rejected and not pursued. Thus, the view of hypotheses is a scientific one: a hypothesis as a verifiable or falsifiable assumption that can be disproved by a single counterexample. For Milo, counterexamples seem to be highly significant in the mathematical discovery process (which also becomes clear at several points in the further course of the interview) and consequently he insists on them. Moreover, Milo's attitude can be characterised as a doubtful one: the counterexample makes him doubt the hypothesis, but first he also doubts the counterexample and checks it once more. It can be said that he takes the role of a supervisor or controller, which also becomes apparent in the further course of the interview. In this way, he ensures the necessary precision and elaboration of the hypothesis.

### 3.4 A short remark on the common mathematical discovery process of Alex and Milo

Since the mathematical discovery process that is previously shown in excerpts takes place in an interactive situation, one cannot disregard the mutual impact that both students have on each other. On the contrary, the interaction of students of different attitudes can bring great potential but also difficulties to their mathematical discovery process. In the case of Alex and Milo, the focus here is on the potential that arises from the interactive process.

Both students contribute to the advancement of the mathematical discovery process. On the basis of their attitudes, the students take a certain role in the process. In the case of Alex and Milo, we see an interplay of both attitudes that has a positive effect on the mathematical discovery process. The attitudes complement each other: Alex's persistent attitude ensures maintenance of the hypothesis by progressively specifying it in response to conflicts that arise. In contrast to that, Milo takes a doubting attitude. Conflicts seem to have a high priority for him so that they make him actually sceptical about the hypothesis. This critical attitude serves as a catalyst for the common mathematical discovery process since it triggers the further development of the hypothesis. By complementing each other, the mathematical discovery process serves as a learning opportunity. Due to the differences in the handling of counterexamples and hypotheses in general, each student individually taken would probably have reached an end beforehand. It is thus the interaction of both attitudes that makes the joint process successful. At the same time, they can learn from each other that the other's attitude in combination with their own helps them to progress in the mathematical discovery process.

## 4 Discussion and conclusions

It was the purpose of this paper to relate the concept of attitudes to students' mathematical discovery process in terms of the handling of counterexamples. In order to answer the research questions, the behaviour of both students was first analysed. On this basis, an attempt was made to draw conclusions about two general attitudes, which the students adopt in the shown excerpt.

1. *What is the behaviour of the two students Alex and Milo when dealing with conflicts (such as counterexamples) in the shown excerpt of the interview?*

By presenting the results of the analysis, it became clear that the behaviour of both students in dealing with the counterexample is fundamentally different. While Alex holds to their hypothesis, Milo becomes extremely sceptical about it and even rejects it. As a way out of conflict, Alex specifies their hypothesis by introducing a classification of staircase types so that the counterexample can be integrated and no longer contradicts the hypothesis.

2. *To what extent can conclusions be drawn from the behaviour about the students' attitudes in dealing with counterexamples during a mathematical discovery process?*

The behaviour of the students can possibly be explained by underlying attitudes that differ in essential points: on the one hand a persistent and on the other hand a doubting attitude. The persistent attitude expresses itself in the defence and maintenance of the hypothesis while the latter rather doubts and contests it.

With respect to the state of research, the counterexample triggers different behaviour at this point. In contrast to the results of Dunbar and Klahr (1989) and Kuhn (1989), the counterexample was neither ignored nor did it lead to a reinterpretation of the evidence. As we could see in the case of Alex and Milo, there are different attitudes that cause different handlings of the counterexample. Concerning Milo, unlike the study results of Kuhn et al. (1988), the counterexample actually has the value of disproving a hypothesis. In his case, this occurs even to such an extent that the second hypothesis would no longer be pursued by him. Alex, on the other hand, takes the counterexample as an opportunity not to reinterpret the evidence as in Kuhn (1989) but to develop the hypothesis further by modifying it.

With regard to a general scientific attitude, both go with some of the desired characteristics in the shown scene. Alex's attitude stands out because of his *confidence*, with which he maintains the previously made hypothesis. In order to resolve contradictions that are contrary to it, he shows a kind of *creativity* that is crucial for problem-solving. In contrast to Alex's, Milo's attitude is characterised by *doubt* and *diligence*. With his way of behaving like a supervisor or controller, he ensures that the joint mathematical discovery process is appropriately accurate and adequately attention is paid to the counterexamples. Due to both students' ability to *cooperate*, the combination of attitudes works like a symbiosis. The challenging mathematical process is thus shared in a kind of cognitive task distribution so that together they act like a mathematician.

Since this paper focuses only on the part of attitudes which become visible in the process of dealing with counterexamples, the ongoing research will further characterize them on the basis of other categories and situations. For instance, the particular role of hypotheses will be further evaluated in this research project. Moreover, a more precise analysis of the mutual impact of students of different and as well of students of similar attitudes is the aim of the study that includes the presented case study. As already mentioned, with regard to the interaction of students of different attitudes, the only case considered here is the one which has a positive effect on the process; it could also be the opposite. Of course, there is also the possibility that the students' attitudes do not influence each other positively, but rather negatively by hindering each other. This can be caused by unfavourable combinations of attitude characteristics.

I conclude that different attitudes in the mathematical discovery process, manifesting in different behaviour, can be gained out of the data. This leads to the hypothesis that the behaviour of students in the process is not arbitrary but influenced by a fundamental attitude, which in interaction with other attitudes, can have a positive or negative impact on the mathematical discovery process. It can thus be stated that for both research and teaching it is worth taking a closer look at attitudes and their impact on mathematical discovery processes. The case of Alex and Milo already gives an insight into diverse manifestations of attitudes. In further research, more case studies will be taken into account to derive concrete and repetitive attitudes, that are consistent over the course of a mathematical discovery process.

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# Examining interpersonal aspects of a mathematics teacher education lecture

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In this paper we present findings from an initial phase of a more extensive study focussed on ways in which prospective mathematics teachers negotiate meaning from mathematics teacher education situations. The focus of this paper is on the language of one mathematics teacher educator and specifically the interpersonal aspects from one mathematics teacher education lecture in Sweden for prospective upper-primary school teachers. We draw on the enactivist view of cognition as a theoretical basis for a methodology we develop that utilises Systemic Functional Linguistics as an analytical tool for studying language-in-use. We exemplify our interpretations through a series of extracts from the mathematics education lecture. This initial phase of our study has exposed several important questions about how participating in an initial teacher education situation may contribute to the development of teacher identities, questions we raise throughout our analyses to provoke further investigation as part of our future research.

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## 1 Introduction

In this paper, we turn our attention to one mathematics teacher educator with a focus on the interpersonal aspects of the mathematics teacher educator's language-in-use during a mathematics teacher education situation. By interpersonal aspects of language-in-use we are specifically referring to "the identities and relationships of the participants in the communication" (Morgan & Sfard, 2016, p. 100). In doing so, we aim to contribute with insights to the research field of mathematics education about how the language of mathematics teacher educators, during teaching situations, may construe the identities and relationships with prospective teachers of mathematics. We highlight a process during a teacher education programme that illustrates how the teacher educator uses past and present experience when talking about mathematics, mathematics education, and prospective teachers' future teaching of mathematics (Ebbelind, 2020). The content of the lecture is then used by the prospective teachers to re-negotiate their ideas of mathematics, mathematics education, and the future teaching of mathematics. In this paper, we report on part of a more extensive study to understand how prospective mathematics teachers negotiate meaning from mathematics teacher educators' language during mathematics teacher education programmes. Understanding this negotiation of meaning addresses the relationship between interaction during teacher education situations and the kinds of meaning realised by the prospective mathematics teachers from those situations. In this way, we are interested in experiences as they are happening which we call pre-reified processes, i.e., what can be observed as happening during teacher education situations that precede and give rise to what others might term beliefs, knowledge, and identity (Ebbelind, 2020; Skott, 2018). A reification process can be described as how lived experiences are represented as something abstract, as reifications. Depending on the perspective used, reifications often include identities, knowledge and beliefs. In this paper, we aim to identify possible ways the mathematics teacher educator supports prospective mathematics teachers in realising meaning during a teacher education situation through analysing his use of language.

We conceptualise mathematics teacher educators as bricoleurs. A bricoleur forms language in each teaching situation, from pre-existing material (for example, research literature or student literature) or past and present lived experiences (for example, as a learner of mathematics or as a classroom teacher of mathematics). During lectures, mathematics teacher educators assemble ideas using whatever experiences come to hand in the immediate social teaching situation.

## 1.1 Background

This study differs from existing research within mathematics education, in three ways: Firstly, in relation to research perspectives and research interests; secondly, in relation to the study of language within mathematics teacher education; and thirdly, in relation to research about mathematics teacher educators. We briefly expand on each of these areas in this section, before discussing our theoretical approach. In the introduction, we mentioned the first way in which this research differs from existing research on beliefs, knowledge, and identity in that our interest is in the pre-reified processes that precede and give rise to what others term beliefs, knowledge, and identity. We focus on processes to reduce the emphasis on objectifications in research about mathematics teacher educators and to have a clear focus on the mathematics teacher educator and the prospective mathematics teachers.

Secondly, this research differs from existing research on the study of language within mathematics teacher education. There already exists a large and evolving body of research focused on the use of language within the domain of mathematics education (Planas et al., 2018), covering topics mainly related to “the language of the learner, the language of the teacher/classroom and the language of mathematics” (p. 198). However, mathematics teacher education and mathematics teacher educators only receive one reference in Planas et al’s. (2018) overview in relation to the use of language, under the title, ‘What more could we learn in the next decades?’ Planas et al. (2018) ask themselves how the methods and research results from the three categories above (the language of the learner; the language of the teacher/classroom; and the language of mathematics) play out during mathematics teacher education situations by mathematics teacher educators. In our view, this question corresponds to a need to pay more attention to issues of language responsiveness in teaching by mathematics teacher educators, to start developing a picture of the kinds of meanings prospective mathematics teachers can realise from teacher education situations in which they engage. Thus, this paper contributes to an identified gap within the field of research on language.

Finally, Beswick and Goos (2018) consider research on mathematics teacher educators, as a general gap within the community of mathematics education research. They define a mathematics teacher educator as “anyone engaged in the education or development of teachers of mathematics” (p. 418). The research community needs to understand mathematics teacher educators’ ways of participating in mathematics teacher education situations. One way to build this understanding is by studying the language of mathematics teacher educators, since mathematics teacher educators guide prospective mathematics teachers in their learning and social development as teachers-to-be. Thus, the language of mathematics teacher educators is an area of research yet to be established, and this research locates itself within this unresearched area. In the following section, we briefly introduce enactivism as the theoretical underpinning of this study, before outlining the methodology (for a more detailed methodological discussion, see Helliwell & Ebbelind, in press).

### 1.2 Enactivism: the theoretical approach

Enactivism is a theory of cognition (learning) that is rooted in biology and viewed from an evolutionary position. In this view, cognition is not a representation of an independently existing world or the construction of an external reality, but rather, it is a continuous adaptive process in which we as individuals co-evolve with our environments (Maturana & Varela, 1998). Learning can be described as “a recursive process linked to actions in the world giving feedback leading to adapted actions" (Brown, 2015, p. 192). Therefore, learning is not seen as a *product* of reification within a context or environment, rather, learning is an active process viewed as dynamic, situated, and emergent (Maheux & Proulx, 2015). We use principles from enactivism to inform our research methodology, and to support and guide our study. Enactivism guides the process described within this paper as a main criterion for research (Gee, 2010) since we view methodology and results as intimately connected.

## 2 Methodology: A recursive inquiry

In this section, we present the methodology that is underpinned by the enactivist view of cognition, the basis of which is a recursive inquiry. In a recursive inquiry, the research process involves "a repeated interaction, with results from one iteration feeding into the next" (Coles, 2015, p. 239). Our research (beyond that reported in this paper) is designed so that each phase of analysis feeds directly into the next phase. A key feature of a recursive inquiry is to acknowledge the relationship between data and the analysis of that data, or in other words, between text and context. Thus, it is important to acknowledge that, as researchers, we are not able to separate ourselves from what we observe, and from the process of data analysis. Therefore, we situate the context of the empirical material alongside our own context as researchers, including our teaching and research backgrounds. In the following section, we outline some of these contextual features before describing how we analyse the empirical material.

### 2.1 The context of the study and the researchers

The empirical material we use in this paper is a transcript of an introductory lecture and seminar for a 30 ECTS (European credit transfer accumulation system) credits (one full semester) mathematics education course. The mathematics teacher educator has been a teacher educator for over 30 years and is well known in Sweden for his academic skills. In this course, he works with prospective teachers to teach upper primary school students (aged 10-12 years) in the context of the reform mathematics movement. The reform mathematics movement "promotes a vision of school mathematics that focuses on students' creative engagement in exploratory and problem-solving activities as they develop their understandings of significant mathematical concepts and procedures" (Skott et al., 2018, p. 164). In Sweden, prospective teachers at these levels educate to become generalists. As a result of this situation, primary teachers will usually teach a variety of subjects. Due to the range of subjects they are expected to teach, their level of education in most these subjects can be modest and their professional motivation is often linked more to the profession as a whole than specific subject disciplines (Ebbelind, 2020).

The community of mathematics teacher educators and mathematics education researchers is a diverse group of individuals from various professional backgrounds and contexts. The authors of this paper are both university-based mathematics teacher educators and researchers in mathematics education. Andreas works at a university in Sweden where he teaches prospective teachers both at pre-school (aged 1-6) level and primary school (aged 7-12) level. He was a pre-school teacher and lower primary teacher for ten years before moving to work at the university as a mathematics teacher educator. His research background links in different ways to Systemic Functional Linguistics, social practice theory and symbolic interactionism. Tracy works at a university in the UK teaching prospective secondary school (aged 11-18 years) mathematics teachers on a one-year postgraduate programme. She taught mathematics in secondary schools for thirteen years before moving to the university as a mathematics teacher educator. Her research background links to the perspective used in this study, enactivism, specifically in relation to the study of mathematics teacher learning and the learning of mathematics teacher educators.

In terms of enactivism as a methodology, Reid (1996) sets out two features of enactivist research: "the importance of working from and with multiple perspectives, and the creation of models and theories which are good-enough *for*, not definitively *of*" (p. 207, emphasis original). As (multiple) university mathematics teacher educator researchers from different cultures and contexts (e.g., Sweden/UK; Primary/Secondary), we consider our different histories of experiences as shaping the ways we each see the world of mathematics teacher education which includes the way we see our data. Thus, in this recursive inquiry, we utilise multiple perspectives by looking at the same data but through different lenses, making multiple revisitations of data using these different perspectives. In relation to creating theories that are good-enough for, not definitely of, we acknowledge the potential for multiple interpretations of the data, and do not claim to be reporting on some external truth of the situation. Rather, we present our interpretations which we invite readers to examine for themselves.

## 2.2 Analysing the lecture

In terms of using enactivist methodology to inform the analysis of language, Coles (2015) describes "five mechanisms that allow an approach to language and learning, consistent with an enactive view" (p. 239). Specifically, these five mechanisms are: recursive inquiry; the systematic search for pattern; equifinality; micro-analysis; and meta-communication (Coles, 2015, p. 239). In this paper, we explicitly exercise two of the five mechanisms (the systematic search for pattern; and micro-analysis), as described briefly below.

According to Coles (2015), the search for pattern involves splitting or segmenting data "in a systematic manner" (p. 239) to identify observable similarities and differences. In the first stage of our analysis, our systematic search for pattern, we use Systemic Functional Linguistics as an analytical tool to split and segment the data so that patterns may emerge that point us to particularly significant moments within the transcript that merit further analysis. Systemic Functional Linguistics serves to uncover, through functional analysis, how the teacher educator produces a particular wording in a specific social practice. Every text reflects that it is about something (ideational meta-function), is addressed to someone (interpersonal meta-function), and uses a particular mode, spoken or written language, for example, to express its meanings (textual meta-function) (Ebbelind, 2020).

In this paper, we focus on the interpersonal meta-function. How the teacher educator is addressing the prospective teachers and other entities that may construe identities and relationships of the participants in the communication. The interpersonal meta-function relates to voice, tense, polarity, and modality. Voice refers to the personal pronoun in the text. Tense refers to whether the proposition is valid for the past, present, or future. Polarity marks if the proposition has positive or negative validity. And lastly, modality relates to the degree of certainty in an utterance (Halliday & Hasan, 1989). We exemplify each of these aspects in the next section. Having used Systematic Function Linguistics as an analytical tool to identify significant moments within the data, we then employ a more detailed 'second stage' of analysis by adopting the micro-analysis techniques as described by Coles (2015). In short, this involves approaching "small sections of transcript with a slow and repeated reading, keeping some questions in mind" (p. 241). The questions that Coles suggests, in keeping with an enactivist view of cognition, are: "What pattern does it follow?", "What pattern does it break?", "What distinction is implied?" (p. 241). It is not the intention in this paper to present a full account of the analysis, but we present four extracts from the full transcript based on the significant moments identified during our systematic search for pattern.

## 3 Analysis and results

In the analysis, we first focused on the voice of the text by marking personal pronouns but, at the same time, marking entities or objects that were evident in the transcript. In the transcript from the lecture, the teacher educator is present through “I”, “my”, “me”, and a “we”. The teacher educator shares the “we” with the prospective teachers. Prospective teachers are present as “you”, “some of you” while teachers in mathematics, as a unit of people, is present through “all” and pupils learning mathematics as “they” and “them”. The teacher educator implies that pupils and some prospective teachers think mathematics learning “is as it is”. The teacher educator relates to the subject of mathematics, collectively, with “many times”, “many students”, “do this”, “it (mathematics)”, “these” and “each other”. Subject voices that are present in the lecture are researchers like “Andrej Dunkel”, “Anna Sfard” and “Governmental reports and steering document”.

Then we marked the tense to highlight if the proposition was valid for the past, present, or future. Many things related to the past in the lecture: the teacher educator being a teacher and teacher educator for a long time, prospective teachers own experience of teaching, being a father and teaching children at home, past reports from the national board of education, experience from being in a classroom teaching, reflecting on deficits in own teaching in the past, the deficit in prospective teachers own past and current experience at the university, and positive experience of being a former teacher and past use of mathematics textbooks as not optional. There are also references to the present: this ongoing lecture, current ongoing mathematics teaching with no understanding, the deficit of not understanding mathematics, students do assignments from the teacher educators past, talking about the experience of this “new mathematics” (expected to be different from their experience), solving problems (expected to be different from their experience), what is mathematics and what is the role of language when teaching today. While most parts in the transcript refer to the past or present, only a few parts relate to the future. These are getting pupils in your (prospective teachers’) future classrooms to think, your responsibility to make things happen, and the future goal of getting pupils to understand (expected to be different from their experience).

Next, we marked the polarity to stress whether the proposition indicated positive or negative polarity. The transcript contained much negative polarity. For example, “not remember”, “not feel”, “not their (understanding)”, “not understood”, “not thinking”, “not understanding”, “not fun”, “not simply”, “not really”, “not do”, “not have”, “not obviously”, “not done”, “not teach”, “not want”, “not explained”, “not know”, “not feel”, “not but”, and “not think”. The use of negative polarity is closely connected to a discursive counterpart, in most cases this counterpart is past experience.

Finally, we marked the modality, which reflects the level of certainty that a clause has. Modality is mostly very high throughout the lecture referring to reform mathematics, family relations, the national board of education, recommendation to the students’ future teaching (strong, “we have to”), critical case from the teacher educator’s past, national mathematics tests, how it should be when teaching (concerning how it should not be), and being ironic about the use of textbooks and governmental investigations of mathematics teaching. However, when the teacher educator talks about the prospective teachers as solving problems, the modality is low. When the teacher educator addresses the prospective teachers implying them to synthesise the content and later make an analogy for students to understand, the modality becomes low. There are also examples of low modality related to mathematics as something for the students to master.

In the systematic search for pattern, we highlighted those emergent patterns from the transcript with its foundation in the analysis above. Here we outline nine observations made:

1. The mathematics teacher educator positions the prospective teachers as a unit, ascribing them all with negative experiences of mathematics.
2. The mathematics teacher educator often goes from past experience to present experience of future teaching.
3. When going from past experience to present experience of future teaching negative polarity is used. The negative polarity is almost exclusively used with negative past experiences of teaching mathematics.
4. Concerning the entities referred to in the text, familiar sources are the experiences of the mathematics teacher educator and the experiences of the prospective teachers.
5. Looking at the tense, we identify that this lecture lacks focus on current and future practices.

6. When focusing on current and future practices, the main parts relate to the deficit story of prospective teachers in relation to mathematics.
7. Modality becomes low only in relation to the prospective teachers and the subject of mathematics. For instance, when the teacher educator talks about the prospective teachers as solving the problem. When the teacher educator addresses the prospective teachers, implying them to synthesise the lecture's content and later, when the teacher educator makes an analogy for the prospective teachers to understand. There are also examples of low modality relating to mathematics as something for students to master.
8. Throughout the lecture, modality is predominantly high, for instance, when referring to content to teach, family relations, the national board of education, the recommendation to the students' future teaching (strong, "we have to"), the critical case from the teacher educator's past, national mathematics tests, how it should be when teaching (concerning how it should not be), being ironic about the use of textbooks and governmental investigations of mathematics teaching.
9. An observable pattern in the transcript is a shift from high to low modality or vice versa.

We will now present the findings from our micro-analysis phase by presenting four short extracts of transcript from the mathematics education lecture, keeping in mind Coles' (2015) suggested three questions: What pattern does it follow? What pattern does it break? What distinction is implied? Even though some of the extracts below contain many of the observed patterns above, we mainly focus on one or two in each extract.

The first extract, extract 1, exemplifies a common theme found in the lecture. When the teacher educator addresses the prospective teachers, the modality is low (e.g., "I think", "do not feel", "was not", "will then try"), otherwise the modality is high throughout the lecture. In the first part of the extract, we can also interpret how the teacher educator starts ascribing the group of prospective mathematics teachers as having had negative experiences of mathematics.

> Extract 1: "One has understanding of things when one does not have to remember what one must remember to be able to know" (Andrejs Dunkels). I think many people here today… who have gone through the whole school system and high school do not feel that way… was mathematics not really something you had to remember … do this here and it will be alright […] Students often do not have the skills needed to be able to present their thinking in writing … It is not

> simply […] how many doors do you have at home? what you come up with, we will then try to bring into this lecture. You should think …

Throughout the lecture, the mathematics teacher educator implies that most (low-modality) prospective teachers have had a negative experience of learning mathematics as students and directly addresses the prospective teachers' previous experiences to promote the reform agenda. We ask ourselves if there is a deficit story here that is non-outspoken, and if that story is consistent throughout the whole mathematics teacher education course (something that we will explore as this research project goes on). An interesting question arising from the analysis concerns the implication of first positioning prospective teachers as students with negative experiences and then aligning them with today's mathematics students, like in the following extract. How do the variety of prospective teachers align with this story? How do the prospective teachers understand the given story?

The following extract, extract 2, exemplifies a common pattern found in the lecture and illustrates a break in pattern concerning modality, from high modality (e.g., "too many students", "we know that") to low modality (e.g., "if you understand", "we want our students", "students often also") back to high modality (e.g., "we must", "you must", "we have to").

> Extract 2: Too many students have not understood anything … We know that from the reports from the national board of education. If you understand, you really do not have to keep such a lot in mind because you know why it is as it is, and you can just pick it up and use it and we want our students to be able to do that in the future. Students often also do not have the skills needed to be able to present their thinking in writing […] We must… you must in the future be able to write mathematically yourself… we have to give students these tools to pass the national tests.

At the beginning of the lecture, we interpret the teacher educator as positioning all prospective teachers within a deficit story. The prospective teachers were grouped into the category "students". What does that mean for the prospective teachers when the teacher educator repeatedly addresses students' experiences during the lecture? In the background, there seems to be a general failure of past teaching of mathematics that is addressed. The failure is used to promote another type of teaching by the mathematics teacher educator.

In relation to this "failure" the teacher educator addresses mathematics as a subject with low modality. The break in patterns here can be observed in the analysis by observing the personal pronoun and the tense. If the tense relates to the current ongoing practice, like extract 3 below, and addresses the prospective teacher (e.g., "you", "we") the modality mostly becomes low. If the tense relates to future teaching practices, like in the last part of extract 2, the modality becomes high. Extract 3 below is an example of low modality, when the teacher educator addresses the subject of mathematics.

> Extract 3: What can it [mathematics] be ...You have to think a lot about this ... It is not that obvious [...] Should we jump into the world of mathematics ... the world that this course is about ... In mathematics, it is not quite as obvious...

Even though the mathematics teacher educator promotes another agenda, namely the reform agenda, the mystification, or exclusivity of mathematics is still a part of the way the lecture is conveyed. One interpretation from the analysis, is that there seems to be a narrative style that can be identified within the transcript, in that there is a story that unfolds. A question this raises for us is how this style influences the prospective teachers while construing the identities and relationships of themselves as teachers-to-be.

The final extract, extract 4, exemplifies a commonly identified pattern. By looking at the tense, one can conclude that this lecture lacks focus on current and future practices. The main parts of the lecture relate to the past experiences of the mathematics teacher educator and the past experiences of the prospective teachers.

> Extract 4: If you do not understand, mathematics is not fun, and it is not so strange really ... so this is connected. I know that I thought it was terribly unfair when I studied mathematics many years ago... because I was a student who did a lot of stuff... did lots of examinations and it went well all the way, but I did not have much understanding of higher mathematics ... I got a completely different experience as a teacher ... when I taught the students ... the students had the same perception and experienced the same as I did ... which I had always experienced and they had passed the courses, but they had not really understood ... Then I really started to think about how to learn to understand ... for real ...

Throughout this lecture the teacher educator promotes the idea that there is another story to tell about teaching and learning mathematics than the expected experiences of the prospective teachers. By observing the analysis of the text there is a kind of anticipation of something to come.

## 4 Discussion

In this paper we have explored the interpersonal aspects of one mathematics teacher education lecture in Sweden since those interpersonal aspects can contribute to construing the identities and relationships of prospective teachers during initial teacher education situations. We focussed on the pre-reified processes (i.e., what can be observed as happening during teacher education situations) to exemplify the potential meanings that may be realised by the prospective mathematics teachers. At the beginning of the lecture, one possible interpretation is that the prospective teachers are positioned within a deficit story. What does it mean for the prospective teachers when the teacher educator frequently addresses their experiences as students during the lecture? In the background, there also seems to be a sense that mathematics teaching has, in the past, been unsuccessful. This failure is used to promote another type of teaching by the mathematics teacher educator who invites the prospective teachers to question their own experiences in relation to the aims of the reform agenda. In doing so, the teacher educator almost exclusively draws on the past experiences of himself and the expected experience of the prospective teachers. The mathematics teacher educator uses his own development as a mathematics teacher as background to promote their change of perspective. We now ask how participating in this initial teacher education situation may contribute to the development of a teacher identity. The process of analysis has led us to asking several questions, it is beyond the scope of this paper to specifically address these questions here, we hope to do this as our study continues.

In this paper we have, with a shared interest, set out to identify possible ways the mathematics teacher educator supports the prospective mathematics teachers in realising meaning during a teacher education situation. In doing so we have used our different research backgrounds. In the next phase of this project, we intend to analyse transcripts from a prospective mathematics teacher attending the exemplified lecture and seminar. This will be done in our pursuit to understand how prospective mathematics teachers negotiate meaning from mathematics teacher educators' language during teaching situations. One broader question that may be of interest within the mathematics teacher education community, is whether mathematics teacher educators arrange their teaching during teacher education programmes with the background of the deficit story of prospective mathematics teachers. How are we, as mathematics teacher educators ourselves, affected by the media debate, that aims to win over the prospective mathematics teachers whose experiences in relation to mathematics may be looked upon as problematic.

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# Emotional classroom climate from a psychological perspective: An analysis of Grade 3 and Grade 6 participant-produced drawings in the context of geometry lessons

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Classroom climate is a rich and an important research concept due to the early discovery of a relationship between positive classroom climate and academic performance and motivation, engagement, participation, and attitude towards school and teaching. In this paper, I focus on the elements of the psychological dimension of emotional classroom climate and the kind of emotional classroom climate in Grade 3 (*N* = 25) and Grade 6 (*N* = 28) school mathematics in the context of geometry lessons by using participant-produced drawings. The students illustrated different elements of the psychological dimension of the emotional classroom climate through physical facial and body features as well as thoughts. Furthermore, the results showed that the emotional classroom climate in both grades was mainly positive, with a negative tendency in Grade 6. The results are discussed not only regarding the research goals, but also regarding their theoretical, practical, and methodological implications.

Keywords: emotional classroom climate, emotions, geometry lessons, participant-produced drawings, primary education

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## 1 Introduction

In recent decades, the study of emotions has gained greater prominence in educational research (Hascher & Edlinger, 2009). During school years, students experience both positive and negative emotions in various subjects (e.g., Reindl & Hascher, 2013; vom Hofe et al., 2002). Among other things, emotions determine the behavior of those involved in teaching (Evans et al., 2009), willingness to learn and to perform and have a strong influence on the mathematical competence growth (vom Hofe et al., 2002). In mathematics education, the topic of emotions has already its field of research (e.g., Dahlgren Johansson & Sumpter, 2010; Laine et al., 2013, 2015; Reindl & Hascher, 2013). For example, the International Comparative Study PISA 2012 analyzed, among other things, emotional orientation in mathematics (Schiepe-Tiska & Schmidtner, 2012). Germany scored slightly below the OECD average in terms of the emotional orientation of enjoyment in mathematics (Schiepe-Tiska & Schmidtner, 2012). Overall, only 39% of 15-year-old female students reported liking and engaging in mathematics because they enjoy it (Schiepe-Tiska & Schmidtner, 2012).

Current research on emotions in mathematics education is predominantly limited to secondary, and less to primary level (Reindl & Hascher, 2013). Yet, there is a decline in enthusiasm for learning and school during the first years of education, and every-day school life is increasingly accompanied by negative emotions (Helmke, 1993). Negatively experienced emotions, such as boredom, are the main accompanying symptoms of school experience (Eder, 2002). Additionally, Reindl and Hascher (2013), reported that positive emotions decrease during the elementary school years, with negatively experienced emotions being subject to a slight recovery effect during the transition from primary to secondary school (van Ophuysen, 2008). These results make clear what significance both positive and negative emotions have for the development in primary school age, and the need and the importance of a stronger focus on primary grades (Reindl & Hascher, 2013).

Whereas the reported studies focused on the individual level of affect, Laine et al. (2013, 2015) expanded the previous work by looking at the interindividual level of affect, namely emotions of/ within a group as a part of classroom microculture of the interactions between the teacher and the students in the context of Grade 3 and Grade 5 mathematics lessons. Even though, previous research on emotional classroom climate focused on mathematics education in general, these studies dealt with (Reindl & Hascher, 2013; Schmude, 2005) or reflected mainly different affect aspects within arithmetic (Laine et al., 2013, 2015). For that reason, geometry lessons were chosen as a study context. Specifically, the main goal of the inquiry presented here was to provide insight into Grade 3 and Grade 6 students' perceptions of emotional classroom climate in the context of geometry lessons by using participant-produced drawings.

## 2 Theoretical perspective

In this section, I first present the construct of classroom climate, with a special focus on the emotional classroom climate. This is followed by the state-of-the-art on emotional classroom climate from an empirical and a methodological perspective. The section ends with two research questions that guided the study.

## 2.1 Emotional classroom climate

A classroom is a social context for learning, which with time develops a distinct social climate or feel (Ashkanasy, 2003). According to researchers (e.g., Eder, 2002; Evans et al., 2009), the classroom climate refers to a shared subjective representation of important characteristics of the classroom. Based on extensive literature review, Evans et al. (2009) defined three complementing components of classroom climate, namely *academic*, referring to pedagogical and curricular elements of the learning environment; *management*, referring to discipline styles for maintaining order; and *emotional*, referring to affective interactions within the classroom. Here, I focus on the last component which can be described through five components: emotional relationship between teacher and students, emotional awareness, emotion coaching, emotional intrapersonal beliefs, and emotional interpersonal guidelines (Evans et al., 2009). According to Götz et al. (2011), emotional climate refers to both positive and negative emotions of a group as well affective attitudes related to the school, people who are associated with the school, areas of specialization, and subjects taught, among others. Evans et al. (2009) argued for the importance of treating emotional classroom climate as a distinct aspect of classroom climate given emotional classroom climate being "superordinate to other classroom climate domains since it interfaces with the conventional academic and management elements of effective learning environments" (p. 131).

The emotional classroom climate can be regarded either from a psychological (i.e., level of the classroom individuals) or a social point of view (i.e., level of the classroom community) (Hannula, 2012). The *psychological dimension*, which is in the focus of the paper, refers to the level of an individual and involves affective conditions, namely emotions and emotional reactions (e.g., fear, joy), thoughts (e.g., "This is difficult."), meanings (e.g., "I could do it."), and goals (e.g., "I want to solve this task.") and affective properties, namely attitudes (e.g., "I like math."), beliefs (e.g., "Math is difficult."), values (e.g., "Math is important."), and motivational orientations (e.g., "I want to understand.") (Hannula, 2012; Laine et al., 2013, 2015). The nature of affective conditions and properties can be classified into three categories, namely positive (e.g., positive emotions such as joy, interest; positive attitude such as "I like mathematics."), negative (e.g., negative emotions such as boredom, fear, anger; negative belief such as "Mathematics is hard."), and neutral (e.g., neutral thought such as "This is a square.") (Laine et al., 2013, 2015; Reindl & Hascher, 2013).

## 2.2 Emotional classroom climate state-of-the-art from a methodological and an empirical perspective

In recent decades, childhood research has experienced a shift from quantitative to qualitative research designs and methods which led to an increased use of participatory, and visual methods and processes in childhood research, such as drawings (Kuzle, 2019), which engage and emphasize children's experiences, perspectives, and understandings making them active agents in the research process (Einarsdóttir, 2007). Furthermore, in contrast to classical data collection methods (e.g., interviews, questionnaires), the use of students' drawings showed significant benefits in qualitative inquiry when working with (young) students (Einarsdóttir, 2007). According to Thomson (2008), and Weber and Mitchell (1995), with visual methods things can be expressed that cannot be easily verbalized, as they require little or no language mediation. This is especially an important aspect when working with young children; it is not easy to get verbally rich answers to questions from young children, since they tend to give monosyllabic answers to questions, they do not consider relevant to them (Hannula, 2007). In addition, they may have difficulties with reading surveys and expressing themselves clearly in writing or within interview contexts due to talking with an often relatively unknown researcher (Hannula, 2007). Furthermore, both methods are – even when using simple scales – particularly time-consuming and accompanied by partially unreliable students' answers (Ahtee et al., 2016; Reindl & Hascher, 2013). As such, these methods have shown not to be always reliable due to participants' young age (e.g., Einarsdóttir, 2007; Pehkonen et al., 2016; Reindl & Hascher, 2013). Kearney and Hyle (2004) found that using participant-produced drawings was more likely to accurately represent participants' experiences, and especially emotions. At the same time, its usage encourages collaborative meaning-making as well as reliable and trustworthy data by establishing a rapport between the researcher and the participant. Such shift in power (im)balance in the researcher-participant relationship with a less researcher-imposed structure has proven to be important when working with primary grade students, especially due to familiarity with the act of drawing, and nonverbal expression (i.e., language mediation, language barrier) at different levels of representation (Ahtee et al., 2016; Glasnović Gracin & Kuzle, 2018). Thus, participant-produced drawings inhibit viewing these with adult eyes (Kearney & Hyle, 2004; Kuzle & Glasnović Gracin, 2020). For that reason, the method is receiving increasing attention in mathematics education research on students' perceptions of classroom climate (Dahlgren Johansson & Sumpter, 2010; Glasnović Gracin & Kuzle, 2018;

Kuzle, 2019; Laine et al., 2013, 2015; Pehkonen et al., 2016).

Laine et al. (2013, 2015) investigated the emotional classroom climate of Finnish Grade 3 ($N$ = 133) and Grade 5 students ($N$ = 136) using students' drawings only. The emotional atmospheres of the classes were classified into five categories (i.e., positive, ambivalent, negative, neutral, unidentifiable) based on the students' and teachers' mode (i.e., facial expressions) as well as on their speech and thought bubbles illustrated in the drawings. In both studies, the emotional classroom climate was mainly positive, with 38% in Grade 3 and 36% in Grade 5. Similar results emerged regarding the ambivalent emotional classroom climate, namely 33% in Grade 3 and 34% in Grade 5. A negative tendency was observed from Grade 3 to Grade 5 with 10% and 14% of drawings, respectively, portraying a negative emotional classroom climate. With respect to using drawings as a research tool, they reported on difficulties interpreting students' drawings only.

Glasnović Gracin and Kuzle (2018) analyzed the emotional climate in school mathematics during geometry lessons using participant-produced drawings (e.g., Kearney & Hyle, 2004). For it, a multiple case study with four high-achieving students from Grades 2 to 5 from the Zagreb area (Croatia) was conducted. The drawings were analyzed based on facial features, and thought and speech bubbles as suggested by Zambo (2006), but expanded by looking also at body language. This was then followed by the holistic evaluation of the emotional climate in each classroom as suggested by Laine et al. (2013, 2015). The results of the study were aligned with those of Laine et al. (2013) with the emotional classroom climate in geometry lessons on the level of the individual being positive (Grade 2 and Grade 3), unidentifiable (Grade 5) or ambivalent (Grade 4), but in no case dominantly negative. Since a multiple case study was conducted, Glasnović Gracin and Kuzle (2018) could not portray a comprehensive picture of the emotional climate in geometry lessons, but rather case-based results. For that reason, the results were neither representative of a large population, nor generalizable.

## 2.3 Research questions

Based on the above theoretical perspective and empirical results, the following research questions guided the study:

1. What elements of the psychological dimension of the emotional climate were reported in the participant-produced drawings of Grade 3 and Grade 6 students in the context of geometry lessons?
2. What kind of emotional classroom climate can be seen in Grade 3 and Grade 6 students' participant-produced drawings in the context of geometry lessons?

## 3 Research process

### 3.1 Research design and subjects

For this study, an explorative cross-sectional qualitative research design (Patton, 2002) using participant-produced drawings (Kearney & Hyle, 2004) was chosen. The research project participants were Grades 3–6 students. In this paper, I report on drawings of 25 Grade 3 and 28 Grade 6 students from different urban schools of two federal states in Germany, namely Berlin and of Brandenburg. Guided by the project experience, Grade 3 students were chosen as students at this age can differ between different types of mathematics lessons, and, thus, can report on their perceptions of the emotional classroom climate in the context of geometry lessons. Lastly, the quality of drawings is already solid to high enough to allow rich insights into the emotional classroom climate. Grade 6 students were similarly chosen for the above-mentioned reasons in addition to being in the last school year of their primary education. Regarding the sampling, from the same school, a maximum of two average students were randomly selected. Typical case sampling as a type of purposive sampling was utilized as a way of collecting rich and in-depth data and to allow for a comparison between other similar samples (Patton, 2002).

### 3.2 Data collection instruments and procedure

The research data consisted of (a) audio data, (b) document review, and (c) a semi-structured interview. The audio data were comprised of the students' unprompted verbal reports during the drawing process, and prompted verbal reports after the drawing process ((a) and (c)). For the document review (b), each student was given a piece of paper with the following assignment: "Dear __________, I am Anna and new to your class. I would like to get to know your class better. Draw two pictures of your mathematics lessons. The first drawing should show what your arithmetic lessons are like and how you view them. The second drawing should show what your geometry lessons are like and how you view them. In each drawing, include your teaching group, the teacher, and the pupils. Use speech bubbles and thought bubbles to describe conversation and thoughts. Mark the pupil that represents you in the drawing by writing "ME". Thank you and see you soon! Yours Anna." (Glasnović Gracin & Kuzle, 2018; Kuzle, 2019). Thought and speech bubbles were used to present children's thoughts as an additional visual representation and to facilitate children's description of their thoughts (Wellman et al., 1996). Here, only the second drawing is of relevance. The students took as much time as needed, usually about 10 to 15 minutes for both drawings. After the students had finished drawing, the drawings were used as a catalyst for a semi-structured interview (Kearney & Hyle, 2004). During the interview, both a free description of the drawing on the part of the child were given (e.g., "Describe your picture to me.") and specific questions based on the child's description were posed (e.g., "How does the child 1, 2, etc. feel in the second drawing?", "What is the reason for that?"). This procedure gave each student the opportunity to frame own experiences, and interpret own drawing. This last part lasted about 5 minutes in total. Multiple data sources were used to assess the consistency, and to increase the validity of the results as was suggested by Einarsdóttir (2007) when employing visual research methods.

### 3.3 Data analysis

As suggested by Patton (2002), multiple stages of the analysis were performed, and contained the following steps: (a) transcribing audio data, (b) analysis of drawings using qualitative content analysis (Patton, 2002), and (c) confirming or adjusting their interpretation by content analysis of the data from the semi-structured interview. Concretely, the author transcribed the audio data (a), and together with another coder coded the drawings independently (b). Here each drawing was analyzed one content category at a time. To examine the emotional classroom climate of each drawing, the individual children drawn were first analyzed, which was followed by the analysis of the illustrated teacher in order to achieve a holistic evaluation of the emotional classroom climate as suggested by Laine et al. (2013, 2015). Concretely, the evaluation was based on both the students' and the teacher's moods as well as on their speech and thought bubbles illustrated in the drawings. According to Koike (1997, cited in Gramel, 2008, p. 36) feelings can be divided into five categories of expression in drawings, namely facial expression, gestures, the facial schema, the representation of situations triggering emotions, and symbols. Here, different facial features, and speech and thought bubbles were analyzed based on the coding manual developed by Zambo (2006), which was expanded with physical body gestures (i.e., body posture, arm position) as suggested by Koike (1997, cited in Gramel, 2008, p. 36), Glasnović Gracin and Kuzle (2018), and Kuzle (2021), to achieve a more accurate representation than was the case in the earlier research of Laine et al. (2013, 2015). In order to facilitate the interpretation of the children's drawings, the semi-structured interviews were analyzed in the same manner (so-called participant-produced drawings) (c). The data from the semi-structured interviews confirmed the coders' analysis of the drawings or added new information than was revealed in the drawings (e.g., emotions of non-depicted students or the teacher, or students and/or the teacher depicted from behind) or on rare occasions gave a completely different picture of the emotional classroom climate. By combining the two data sources, the consistency of the results was assessed which consequently increased the validity of the results as was reported in similar studies (e.g., Kearney & Hyle, 2004; Kuzle, 2021; Kuzle & Glasnović Gracin, 2020).

Following the rating of the children drawn, the holistic evaluation of the emotional classroom climate in the context of geometry lesson was assessed by combining Zambo's (2006) rating, and Laine et al. (2013, 2015) emotional classroom climate categories. If a child's rating of a category was emotionally positive, a counter (+1) was noted. If the assessment was negative, a negative counter (-1) was noted, and if the assessment was neutral, the symbol 0 was noted (Zambo, 2006). If none of the categories was drawn, it was classified as unidentifiable and received a dash (-) (see Table 1). In that manner, the ratings from +2 to -2 for the entire drawing were possible. After rating each feature, the "counters" were balanced against each other. If the score was 0, the emotional state of the respective child was rated as neutral; if the score was positive, it was rated as positive; and if the score was negative, it was rated as negative. If an individual contained both positive and negative characteristics, it was coded as ambivalent. As can be taken from Table 1, the emotional feeling of child 1 was coded as negative since counters for physical facial features as well as for speech/thought bubble features were assigned each -1. Following the rating of the children drawn, a slight adaptation of Laine et al. (2013, 2015) emotional classroom climate categories were employed for the purposes of the holistic evaluation of the emotional climate as was earlier reported by Kuzle (2021). The emotional categories were as follows: positive (i.e., persons smile, think or behave positively, although some of the expressions can be neutral), ambivalent (i.e., there are both positive and negative facial/body language expressions or thoughts in the drawing), negative (i.e., persons are sad or angry or think/behave negatively, although some of the expressions can be neutral), neutral (i.e., all facial/body language expressions or other thoughts are neutral), and unidentifiable (i.e., no facial/body language expressions or thoughts are present in the drawing) (Laine et al., 2013, 2015). If identifiable and non-identifiable persons were illustrated, only the non-identifiable ones were identified in the overall image analysis but were scored as neutral.

**Table 1.** Exemplary coding of the emotional feeling of the drawn child.

| Child | Physical and speech/thought bubble features | Feature clues | Explanation | Score |
|---|---|---|---|---|
|  | Face features | Mouth | - | - |
| | | Eyes/eyebrows | Closed, downward slant | -1 |
| | | Face drawn symbols | - | - |
| ”me” | Total: Physical face features | | | -1 |
| | Body features | Arm position | Downward | 0 |
| | Speech/thought bubble features | Symbols | - | - |
| | | Signs | - | - |
| | | Words | “I find geometry hard.” | -1 |
| | Total: Speech/thought bubble features | | | -1 |
| Total: Child 1 | | | | -2 |

Two researchers coded the students’ data separately from one another. The inter-rater reliability was high (90% agreement). Nevertheless, we discussed the differences in coding taking into consideration both students’ products and refined at the same time the coding manual. This decision mainly related to the drawings in which the protagonists were depicted from behind or in an extremely simplified or generic manner. Furthermore, there were a few disagreements regarding the nature of individual thought features, such as “good”, “okay” which were then discussed. Also, it was also agreed that the final decision about the nature of a counter assigned to a particular physical feature would be based on the interview data. Such inconsistencies were primarily seen in Grade 3 students’ drawings. Due to analyst triangulation, adjustments were subsequently made to our coding, after which the interrater reliability was 100%, and the same time contributed to the verification and validation of the qualitative analysis. Afterwards, descriptive statistics were calculated in order to determine the kind of emotional classroom climate.

## 4 Results

This section is divided into two parts. In the first part, the focus lies on different elements of the psychological dimension of the emotional classroom climate and their nature in both grades (i.e., similarities, differences), whereas in the second part on the kind of the emotional classroom climate in Grade 3 and Grade 6.

### 4.1 Psychological dimension of the emotional climate in Grade 3 and Grade 6 participant-produced drawings in the context of geometry lessons: Similarities and differences

Tables 2 and 3 illustrate psychological dimension of the emotional classroom climate on the basis of the physical features (e.g., face, body), and speech and thought bubbles reported in the participant-produced drawings which were assigned one of the three categories (i.e., positive, negative, neutral).

**Table 2.** Nature of different emotions illustrated in Grade 3 students' drawings in the context of geometry lessons.

| **Feature and thoughts** | | **Emotional classroom categories** | | |
|---|---|---|---|---|
| | | **Positive** | **Negative** | **Neutral** |
| Physical face features | Eyes/ eyebrows | Wide open; upward slant | Closed; downward slant | Typical without expression; no slant no special features |
| | Mouth | Full, wide smile | Angry; open in a scream; drawn as a jagged line; portrays a frown | Drawn as a straight line |
| | Symbols | – | Tears; tongue stuck out | – |
| Physical body features | Arm posture | In the air (open upwards); request to talk | – | In action; open downwards; on/behind the back; on the table |
| Thoughts | Symbols | Hearts; peace sign | !!! | – |
| | Signs | Laughing smiley | Smiley with slanted mouth | Smiley w/straight mouth |
| | Words | "I am in a good mood."; "Yes!"; "AAAAA"; easy; fun; cool; interesting; I like/love geometry; very happy; it feels good | "boring"; "I find geometry difficult. | That's a ... |

From both tables, similarities and differences can be observed regarding the features and thoughts that were illustrated as well as their nature. Regarding the neutral emotional classroom category, no differences could be observed regardless of the feature and thought. The same applies for physical face feature "eyes/eyebrows" across all three emotional classroom categories. Whereas physical face feature "mouth" was the same for positive and neutral emotional classroom categories, only Grade 3 students' drawings revealed negative features, namely angry mouth, screaming mouth, mouth turned downward. Thus, physical face features reflected different positive (i.e., joy), negative (i.e., anger, sadness) and neutral emotions and emotional reactions. Joy was illustrated for instance with wide open eyes and mouth, sadness with a mouth portrayed as a frown and tears, and anger with mouth portrayed in a scream or with a tongue stuck out. The physical body feature "arm posture" was the same for positive and neutral emotional classroom categories in both grades, but Grade 6 drawings revealed two negative features, namely arms crossed on the body and holding/playing with a smartphone. Both reflect an emotional reaction whereas the former is a sign of discomfort, uneasiness or insecurity, and the latter a sign of boredom or disinterest.

The students of both grades illustrated different emotions and emotional reactions, thoughts, attitudes, and beliefs using symbols, signs, and words. Here, Grade 3 students used these to illustrated positive emotional classroom category (i.e., hearts, peace sign), and Grade 6 students to illustrate negative emotional classroom category (i.e., zzz, dark scribbles). The former reflects positive emotion of affection and optimism, whereas the latter negative emotion of boredom and anger. As mentioned, signs were also used to illustrate positive emotion of joy (i.e., smiley) as well as negative emotion of anger (i.e., child fighting with a sword, crumpled book). Lastly, both groups of students used words to communicate their thoughts, attitudes and beliefs about geometry lessons. Here, Grade 3 students' drawings revealed more positive statements than Grade 6 students' drawings. Similarly, Grade 3 students' drawings revealed fewer negative statements than Grade 6 students' drawings. Positive attitude towards geometry came from two Grade 3 students only ("I like geometry.", "I love geometry."). A negative belief about geometry came also from one Grade 3 student by saying "I find geometry difficult." Most often Grade 3 students used words to convey positive thoughts about geometry lessons (e.g., easy, fun, "Yes!"). On the other hand, Grade 6 students used words to convey negative thoughts about geometry lessons (e.g., "Oh no!", "Not again.", too hard, too difficult, confused, blah blah). In other word, the drawings of Grade 3 students revealed more positive thoughts in the form of words in their drawings than those of Grade 6 students. Or, Grade 6 students revealed more negative thoughts in the form of words in their drawings than those of Grade 3 students. The interviews revealed that different aspects of affect – both positive and negative – were due to the teacher's teaching practices, the content, and the working method.

**Table 3.** Nature of different emotions illustrated in Grade 6 students' drawings in the context of geometry lessons.

| **Feature and thoughts** | | **Emotional classroom categories** | | |
|---|---|---|---|---|
| | | **Positive** | **Negative** | **Neutral** |
| Physical face features | Eyes/ eyebrows | Wide open; upward slant | Closed; downward slant | Typical without expression; no slant no special features |
| | Mouth | Full, wide smile | – | Drawn as a straight line |
| | Symbols | – | – | – |
| Physical body features | Arm posture | In the air (open upwards); "Me, me, me" (request to talk); pointing at something | Crossed arms on the body; holding a smartphone | In action; open downwards; on/behind the back; on the table |
| Thoughts | Symbols | – | Zzz; dark scribbles | – |
| | Signs | Laughing smiley | Child fighting with a sword; crumpled books | Smiley w/straight mouth |
| | Words | "Almost done.!; "That's easy.!; "I understand it well." | blah, blah; confused; (too) hard; really complicated, "What is she babbling about?"; "All of them?"; "The faster I finish, the faster I can read."; "Oh no!"; "Not again!"; "What?!"; "Always just writing!" | That's a … |

## 4.2 Emotional classroom climate in Grade 3 and Grade 6 participant-produced drawings in the context of geometry lessons: Similarities and differences

After analyzing the physical features (e.g., face, body), and speech and thought bubbles of drawn children and the teacher in the drawings, they were classified into five categories (i.e., positive, negative, ambivalent, neutral, and unidentifiable) (Laine et al., 2013, 2015). In Table 4 the results regarding the emotional classroom climate in Grade 3 and Grade 6 drawings in the context of geometry lessons are presented.

**Table 4.** Absolute and relative frequencies of the reported emotional states in the context of Grade 3 geometry lessons.

| | Emotional classroom climate categories | | | | |
|---|---|---|---|---|---|
| | Positive | Ambivalent | Negative | Neutral | Unidentifiable |
| Grade 3 students (*N* = 25) | 15 (60%) | 6 (24%) | 1 (4%) | 2 (8%) | 1 (4%) |
| Grade 6 students (*N* = 28) | 13 (46%) | 12 (43%) | 0 (0%) | 2 (7%) | 1 (4%) |

In total, 60% of drawings (*n* = 15 drawings) of Grade 3 students represented the emotional climate in the context of geometry classroom as positive as opposed to 46% of drawings (*n* = 13 drawings) of Grade 6 students. Thus, a bit less the half of the Grade 6 students perceived the emotional classroom climate as positive. Nevertheless, the difference of 14% between both grades is not significant. On the other hand, ambivalent emotional classroom climate was reported in 43% of Grade 6 students' participant-produced drawings (*n* = 12) as opposed to 24% of participant-produced drawings of Grade 3 students (*n* = 6). Thus, the percentage of drawings portraying ambivalent emotional classroom climate in Grade 6 differed minimally in percentage of those portraying positive classroom climate. The difference between both grades, however, was in this case significant. In both grades, the interviews were somewhat aligned with the data from the drawings but revealed more negative features than the drawings since on occasions the students or the teacher were portrayed from the behind.

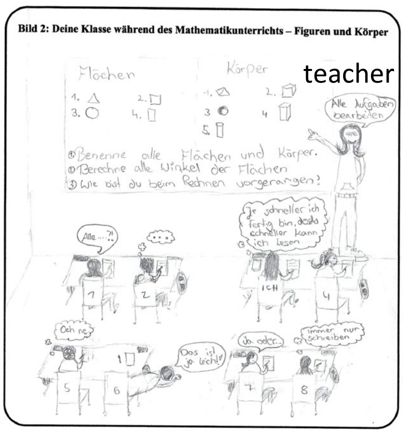


Figure 1. An example of an ambivalent emotional classroom climate from a Grade 6 student.


Figure 2. An example of a negative emotional classroom climate from a Grade 3 student.

The drawing shown in Figure 1 is an example of a Grade 6 student drawing that was rated as ambivalent since both positive (e.g., teacher and child 4 smiling, “That’s easy.”), and negative features (e.g., “Oh no.”, “Always just writing.”) were illustrated with some neutral ones (e.g., “Do all the tasks.”). As opposed to one drawing of Grade 3 student portraying a negative emotional classroom climate, no drawing of Grade 6 student portrayed such climate. The drawing shown in Figure 2 is an example of a drawing that was rated as negative since only negative features (e.g., child 3 crying, child 1’ and 3’ mouth portrayed as a frown, teacher’s mouth open in a scream) with some neutral ones (e.g., arms closed downwards) are illustrated. The interview revealed that the mood was determined by a quarrel between the students which consequently influenced the teacher’s mode. In both grades, the percentage of drawings illustrating neutral or unidentifiable emotional classroom climate was similar or the same. With respect to the later, there are no facial or body expressions, and speech and thought bubbles could be identified. Children’s names are written down on drawn rectangles, which most likely represent desks. The interviews did not provide any further information.

## 5 Discussion and conclusions

In this study, participant-produced drawings were used as a data source for researching holistic primary grade students’ perceptions of emotional classroom climate during mathematics in the context of geometry lessons. To express different aspects of the psychological dimension of affect (e.g., emotions and emotional reactions, thoughts, attitudes, beliefs) as well as their kind (i.e., positive, negative, neutral), the students used various physical features of the face (i.e., eyes/eyebrows, mouth) and thoughts (i.e., symbols, signs, words), but less physical body features (i.e., arm posture). The analysis of Grade 3 and Grade 6 children’s drawings and interviews revealed a positive teaching climate in the context of geometry lessons. However, in Grade 3 such emotional classroom climate predominated (more than 50% of drawings) which was not the case in Grade 6. Furthermore, a striking high percentage of drawings illustrating an ambivalent classroom climate in Grade 6, which was accompanied by a lower percentage of positive ones emerged from the data. Similar results were reported in a study by Dahlgren Johansson and Sumpter (2010) where the majority of Grade 2 students expressed positive attitudes towards mathematics and connected mathematics to a positive feeling, whereas a decrease in positive emotions in Grade 5 compared to Grade 2 was reported. This is also aligned with the results of

Reindl and Hascher (2013) that reported on a negative trend in the course of the primary grade school years regarding the emotional experience, which was also observable when comparing both studies of Laine et al. (2013, 2015). A possible explanation for this finding could be a child's optimism, which is much more pronounced in younger students than in older ones (Hasselhorn, 2005). Since these studies focused on mathematics lessons in general, but mainly used items pertaining to arithmetic or students illustrated arithmetic lessons, it may be that this trend is independent of the mathematics subfield. Grade 3 students' data revealed more positive conditions and properties in geometry lessons than was the case with Grade 6 students. Especially worrying is the nature of different negative emotions, namely boredom, fear and anger, and the way these were illustrated since these have been recognized as negative influencing factors regarding mathematical competence growth (vom Hofe et al., 2002).

The study results confirmed to some extent the results of Laine et al. (2013), when both positive (38%), as well as ambivalent drawings (33%), are considered jointly. These results may be also due to study conditions since the collective emotional atmosphere was researched, which may have contributed to somewhat skewed results. Furthermore, the study looked at mathematics lessons not focusing on specific mathematical content. Taken the experience in the project, students mostly associate mathematics with arithmetic and have more difficulties learning arithmetic content than geometry content. This may have contributed to differences in both studies' results when both categories are treated separately. This assumption is aligned with Krauthausen (2018), and Radatz and Schipper (1983) who state that geometry due to its alternative teaching concepts (e.g., action-oriented instruction, discovery learning) may promote positive mathematics-related affect. Taken that the results did not entirely confirm the results from the earlier research (e.g., Laine et al., 2013), and geometry lessons were chosen as a study context, the next possible step may be to contrast the emotional classroom climate between arithmetic and geometry lessons with a special focus on the specificities of these two mathematics subfields in connection to students' (perceptions of) emotions.

The use of participant-produced drawings allowed interpreting the meanings that the students had given to the situations and objects they had presented which would not have been possible using quantitative methods. Thus, the drawings which were triangulated with the interviews (i.e., participant-produced drawings) allowed an in-depth understanding of what each child had drawn, and to more accurately represent their emotions and perception of the emotional classroom climate. Especially, the interviews gave also an insight into the teachers' mode, and pedagogical skills in geometry lessons. These had either a positive or negative influence on the children's emotional experience in the context of geometry lessons. Given that the teacher is considered as an important factor of the perceived emotions in the classroom, it is a relevant factor in determining the emotional classroom climate (Evans et al., 2009). This provides another interesting research direction, namely to examine the interaction or the influence of psychological dimension (level of the individual) between or on the social dimension (level of the community) of mathematics-related affect levels which is still a rather unexplored area of research.

This study was an exploratory study with a rather small sample, and for that reason cannot be generalizable. Nevertheless, since purposive sampling was used, the results are representative of other similar samples. Futures studies involving a larger data sample and/or using other sampling methods (e.g., maximum variation sampling, probability sampling) could contribute to generalization of the results to a population. The results of this cross-sectional study showed some evidence of increasing negative elements of the psychological dimension of classroom climate from Grade 3 to Grade 6. A longitudinal study from the beginning of school to the transition to secondary school of each individual reference group could be aimed at to investigate the course of the emotional climate in the classroom. Lastly, working with the entire classroom or schools may provide a more holistic insight into the collective or school emotional climate in primary school mathematics.

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