LUMAT
International Journal on Math, Science and Technology Education

Research article

Beyond the dichotomy: Deeds and explorations as a mutually enhancing dyad in undergraduate mathematics education

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Abstract: Students' development of routines is essential to learning, particularly in STEM studies where there is a goal of preparing students for directly applying mathematics for practical ends. We hypothesize that this impacts on how they engage with routines. We report from an empirical study which explores different types of routines that undergraduate students engage in when using an animation tool to model motion in a real-world context. Drawing on commognitive theory, we conducted a fine-grained qualitative analysis of the discursive activity of one group of three engineering students. Rather than aiming for a broad generalization, the study uses this dataset to examine the mechanisms through which students shift between deed-oriented and explorative routines. The analysis shows that these routines do not function as separate forms of engagement, rather they developed as a mutually reinforcing dyad – a dyadic relationship. We argue that recognizing this relationship provides a refined lens for understanding how environments may support possibilities not only for practical learning, but also for discursive learning.

Keywords: commognitive framework, explorative routines, deed-oriented routines, animation tool

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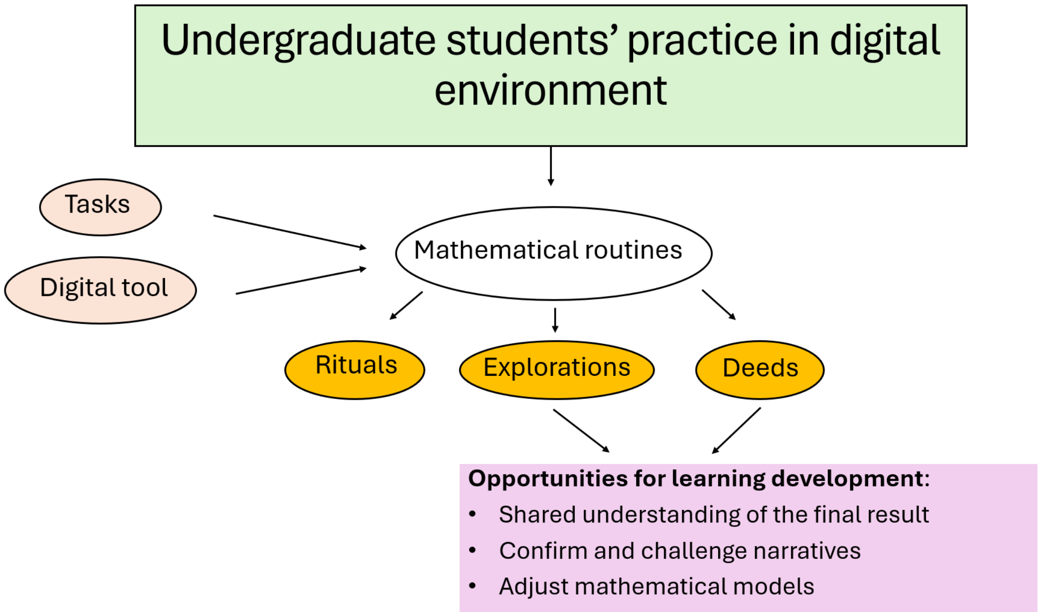


Received 24 August 2025 / Accepted 11 December 2025 / Published 19 December 2025
Published by the University of Helsinki, Finland and LUMA Centre Finland.

1 Introduction

For many students in STEM fields, the purpose of their mathematics studies is to prepare them for applying mathematics for specific practical ends. In engineering, for example, they are expected to model physical processes, such as movement of mechanical objects, and numerical optimalization in machine settings. Research shows, however, that there is a gap between how mathematics is taught in engineering studies and how mathematics is actually used by engineers in the workplace, ignoring the specific needs of engineers (González-Martín et al., 2022; Maass et al., 2019). To tackle the gap between educational and professional practices, this paper presents results of research examining the mathematical practice a group of students engage in while operating on mathematical and physical objects using a digital animation tool.

For STEM students, the mathematics is often connected to physical phenomena, where the end result is of greater importance than proving mathematical theorems or telling other stories about mathematical objects which are of importance for mathematicians. Thus, we hypothesize that the two student groups are involved in different practices when operating on mathematical objects. In other words, what students tend to repeat when they encounter certain familiar situations might differ between disciplines. According to Sfard (2008) students develop meanings to their actions by "reproducing familiar communicational moves in appropriate new situations" (p. 195). In this paper, we investigate the patterns within students' discourse that repeat themselves in familiar situations, and the relations between such patterns. From a commognitive perspective on learning, such patterns are called *routines*. Different types of routines are divided by their focus, either on the process itself or on the product which is the outcome of the process. Their relation is of particular interest, as engineers often adjust and modify their mathematical models based on observed obtained outcomes (Gainsburg, 2006). If the desirable outcome is not provided, they go back and adjust their original models, making the students involved in an iterative process (Stillman et al., 2020). Research on routine use has mainly been concentrated on how students move from a focus on alignment with the teacher (do what the teacher does) towards focusing more on exploring the mathematics operated upon (tell stories about mathematical objects). Less researched is the interplay, in this move, between a focus on the final outcome in terms of physical or mathematical objects (deeds) and in terms of constructing mathematical narratives (explorations). We will elaborate more on routines and their development in the section 'Theoretical background'.

Several studies have focused on different types of routines and how students navigate between them. For example, Viirman and Nardi (2019) have paved the ground for further research on this topic. They report on undergraduate students' engagement with different routines within the disciplines of biology and mathematics. One of their results is that explorative routines (telling stories about mathematical objects) mainly took place within biological discourse, whereas engagement in the mathematical discourse was mainly ritualized (mimicking a more knowledgeable other, such as the teacher). Also, the students' routine use in one discourse impeded their routines in the other discourse, in correspondence with the results of previous reports from the same project (Viirman & Nardi, 2017; 2018). At the university level, students often navigate between different disciplinary discourses. For example, in engineering education, mathematical objects, such as integrals, are often introduced using elements from both mathematics and engineering (González-Martín, 2021).

Kontorovich (2021) is concerned with the question of *when* routines are used (in what circumstances they are used in relation to which tasks) in studying a class of 18- and 19-year-old students' discourse regarding square roots. The study shows how the students' discourse is affected by the tasks and explains how the tasks may be interpreted differently by the students and by the teachers. Biza (2021) investigates where routines originate through analyzing responses to a questionnaire about the mathematical object tangent line by 182 mathematics undergraduate students at two universities, all enrolled in a first-year calculus course. She introduces the notion of *discursive footprint* which is to be understood as how a mathematical object, in this case the tangent, is used across mathematical domains. Other researchers focus on designing tasks that support a particular routine use, in particular explorative routines (e.g., Cooper & Lavie, 2021). Baccaglini-Frank (2021) shows how dynamic interactive mediators can foster high school students' explorative routine use in mathematical discourse. Our goal is to study the role of deeds in students' use of mathematical routines, where the students attempt at transformation or re-arrangement of physical and mathematical objects in a real-world context. Our aim resulted in two research questions:

RQ1: What characterizes the students' routine use in the task situations?
RQ2: In what ways are relations between deeds and explorations revealed in the students' interaction?

2 Theoretical background

2.1 Routines: deeds, rituals and explorations

We take a commognitive perspective on learning, where learning is perceived as a process of routinization of students' actions (Sfard, 2008). In the study of learning, students' routines, perceived as "repetition-generated patters of our actions" (Lavie et al., 2019, p. 153), are the basic unit of analysis. When we meet a familiar situation, we tend to repeat what we have previously done or seen done by others. In daily life, we usually greet new people with a handshake while introducing ourselves by name. In mathematics classrooms, students learn procedures of how to act (e.g., how to integrate functions) and when the procedure (e.g., integration) is suitable. When students meet a familiar task, they tend to do what they have previously done in similar task situations or have seen done by others (e.g., the teacher or a more knowledgeable classmate).

Each routine consists of three parts: initiation, procedure and closure. A procedure to be performed is evoked under some conditions (relates to the initiation-part of a routine) and when the procedure is considered complete relates to the closure of a routine. The parts initiation and closure are the *when* of the routine and the procedure is the *how* of the routine (Sfard, 2008). Sfard distinguishes between explorative and ritualized routines. Explorative routines, or simply explorations, aim to produce mathematical narratives (stories about mathematical objects) that can be endorsed or rejected by the mathematical community. The produced narratives may be new to the learner (but known to the mathematical community) or to the mathematical community at large. For example, if a student is asked to find the definite integral of a velocity function, she might proceed using integration procedures to form a narrative about the new integrated function as distance travelled. Rituals, on the other hand, are performed for the sake of being in social alignment with others in the mathematical community. The question guiding the learner is "How do I proceed?" A ritual may be simply an imitation or a reproduction of someone's former performance. The question of *why* the procedure works might not yet lay within the learner's grasp; rather it is performed because it makes sense to others. For example, a student who is asked to find the definite integral of a velocity function might proceed by following in the footsteps of her teacher without being aware of how this procedure can be endorsed. She is more focused on doing what she has been told (or shown) and to be in social alignment with others, and thus participates in a more ritualized way. Her aim is carrying out specific steps previously learned, rigidly following previously known procedures, while fulfilling goals set by other participants in the discourse, e.g., a teacher (Sfard, 2008). To methodologically determine whether a particular routine is ritualized or explorative, one needs to attend to the initiation, procedure and closure of the routine. For instance, was the student determined to reproduce the calculations of the teacher or a peer-classmate? Or was the student rather choosing between several procedures as dictated by the needs of the given task situation? (Sfard, 2008). In summary, rituals are characterized as process-oriented, while explorations are characterized as product-oriented (Lavie et al., 2019).

The routines discussed so far are discursive, in that they are performed for discursive purposes. Sfard (2008) also considers routines that are performed to achieve change in objects, either discursive or primary (objects existing independently of discourse). Such routines, called *deeds*, are the focus of this paper. An example of a deed is a cashier counting change to leave the correct amount of money back to a customer. The cashier is not endorsing narratives about counting, but rather making a fair money transaction. When a student in a mathematics classroom is focused on getting a particular answer, e.g., by manipulating the terms of a velocity function in order to get a certain answer already known to him, his routine use is more deed-oriented. In the latter case, the object to be obtained is discursive, while in the example with the cashier, the obtained object is primary (i.e., coins) (Lavie et al., 2019). Generally, deeds are limited in their use due to the strong connection to the visual mediators, i.e., physical objects or artifacts mediating mathematical meaning, present in the task situations the students are engaged in (Sfard, 2008). In fact, one of the greatest problems students have with doing mathematics, according to Nunes and Bryant (1996), is to "understand that mathematical relations and symbols are not bound to particular situations" (p. 248). This challenge occurs especially in the performance of deeds, where the performance is connected to the visual mediators present in the situation. Table 1 summarizes the three types of routines.

Table 1. A short summary of the three routines

Routine type	A short description of routine
Rituals	Being in social alignment with others.
Explorations	Producing mathematical narratives.
Deeds	Arranging/re-arranging primary or discursive objects.

2.2 Routine development, and development of the notion of routine

Early commognitive research have examined how learners enter mathematical discourse through participation in routines. For example, Sfard and Lavie (2005) showed how young children compared boxes of marbles through deed-oriented routines. Their participation was centred on physical objects (boxes) rather than narratives.

Once the children begin to count the marbles in order to determine which box contain the most marbles, their participation becomes more explorative, still with an element of a deed (at the end they will choose a box, but the choice is mediated by counting). Participation "may start as a deed but then become an exploration as the student begins to investigate mathematics associated with the physical activity" (Wood, 2016, p. 331). For example, students begin to arrange a geometrical shape into a new geometrical shape and continue to explore the relationships between these geometrical shapes (Wood, 2016). In this paper, we want to investigate a potential interplay between deeds and explorations.

The children in the above example initially engaged in practical routines, deeds, but they did so in a ritualized manner. However, their actions are expected to eventually become discursively mediated and become explorations (Sfard & Lavie, 2005). Rituals are perceived as a necessary step towards explorations, as learners need to imitate what more knowledgeable others have done before they can begin to explore mathematical objects. Hence, rituals may act as an 'entry ticket' to new discourses, and as germinal routines they are essential at the early stages of routine development.

In early commognitive studies, routines were named after the tasks they were meant to accomplish, e.g., the *routine of numerical comparison* analysed in Sfard and Lavie (2005). A routine was perceived as a constantly evolving pattern for implementing a particular task. However, over the last decade, commognitive research has developed a deeper understanding of the notion of routine. For instance, recent research (e.g., Biza, 2021; Lavie et al., 2019; Viirman & Nardi, 2021) has paid greater attention to how task setter and task performer may interpret the task differently. In an attempt to address this and other similar concerns, Lavie et al. (2019) provide an operationalization of the notion of routine: “a routine performed in a given task situation by a given person is the task, as seen by the performer, together with the procedure she executed to perform the task” (ibid, p. 161). Here, a task situation denotes “any setting in which a person considers herself bound to act” (p. 159), and a task “as understood by a person in a given task situation, is the set of all the characteristics of the precedent events that she considers as requiring replication” (p. 161). It follows that the notion of routine is context dependent. In other words, a routine is connected to particular task situations where the students are obligated to perform a procedure. How and when the student performs this procedure is the main focus in studying routine use to determine its characteristics. Still, even as each person develops his or her own routines, these routines are inspired by what other more experienced performers do, leading us to perform in similar ways (Lavie et al., 2019).

Furthermore, Lavie et al. (2019) acknowledge that routine use is rarely purely ritualized or purely exploratory. Rather than being viewed as a dichotomy, ritual and exploration should be considered as the opposite ends of a spectrum, where any routine use typically involves both ritualized and exploratory elements. Lavie et al. (2019) also discuss the role of deeds from this perspective, stating that since they are also concerned with outcome rather than process, they have more in common with explorations than rituals. This suggests the possibility of a ritual-deed spectrum, similar to the one for rituals and explorations, but where the end-goal is change in objects rather than in narratives.

Following the work of Lavie et al. (2019), a growing number of studies have been researching ritualized and explorative participation in teaching and learning practices at schools and universities (Nachlieli & Tabach, 2019; Viirman & Nardi, 2019). Viirman and Nardi (2019) found the perception of a “dichotomous binary” (p. 233) between the two unhelpful in studying undergraduate students’ navigation between the discourses of biology and mathematics. Students’ assumption making in modelling activities was characterized by a fluid interplay between explorations and rituals. Viirman and Nardi also found evidence of ritualized routine use paving the way for more explorative engagement in a complex interplay between the two.

For learners to be able to act in new situations, it is anticipated that they need to first follow in the footsteps of others (Nachlieli & Tabach, 2012; Sfard, 2008; Sfard & Lavie, 2005). Routine use then needs to progress through a *de-ritualization process* in which the performers’ attention gradually shifts towards the outcome of the performance rather than the performance as such (Sfard, 2008). Such de-ritualization is typically a slow and gradual process (Heyd-Metzuyanim et al., 2018; Lavie & Sfard, 2019; Sfard & Lavie, 2005). To identify shifts from ritualized to explorative routines in the de-ritualization process, Lavie et al. (2019) provide a partial list of changes indicating such shifts. In the process, the routine use becomes more *flexible* (a certain procedure is enacted for a range of tasks or various procedures might be enacted to solve a given task), *bonded* (each step carried out in the procedure is used as an input to the next step), *applicable* (the procedure is applicable to solve the task), *agentive* (student’s freedom of making decisions in initiation of a performance, how to proceed, how to continue and when to stop and evaluate the performance increases), *objectified* (the storytelling becomes more abstract and the procedure is derived from the properties of the mathematical object) and *substantiated* (the student argues and justifies that the narratives hold and can be regarded as 'truths' about mathematical objects). A shift towards greater de-ritualization may express itself in the strengthening of at least one of these six characteristics (Lavie et al., 2019).

Research has so far mainly emphasized the roles of rituals and explorations (Nachlieli & Tabach, 2019; Tabach & Nachlieli, 2015; Viirman & Nardi, 2021), while the role of deeds has remained underexplored. In particular, there is a relative lack of research on the relation between deeds and explorations, and how to operationalize these routines in data. In an aim to contribute to this line of research we therefore, in this paper, provide an analytical tool to investigate deed-oriented and explorative routines, as described in 'Methods'. We discuss in particular the relation between deeds and explorations to find possibilities for deeds to enhance explorations and vice versa by scrutinizing the path between these two types of routines.

This paper contributes to the existing literature about students' routines, especially on deeds where research is still scarce. We are not addressing STEM students' practices in general, rather STEM education is a context for the research and influenced the tasks presented to the students.

3 Methods

The study reported in this paper derives from an ongoing research project whose overall aim is to characterize first year students' discourse about functions while solving tasks together using a digital animation tool (see Figure 1). Here, we focus on students' routine use, and in particular on their engagement with deeds. We explored students' engagement in an integral-area context with a presence of a digital animation tool. Four task situations were designed with a focus on students' engagement with the topic of integration. The students' engagement was analysed from a commognitive perspective, that enabled us to distinguish the different kinds of student engagement (ritualized, explorative or deed-oriented) and if their engagement changed across the task situations. In our analysis, we have included all three types of routines as they appear in our data, but our main focus will be on the relation between deed-oriented and explorative routines as the students alternate between the two.

3.1 The task situations

The students were presented with four task situations in a written format and a digital animation tool called Sim2Bil on a laptop. The task situations support students' realization of the definite integral as an algebraic operator, an area under a curve and an area-accumulation function. The digital tool affords an animation of two cars, one red and one green, running in a straight line from a start line to a finish line (upper left corner in Figure

1, which shows the interface of the tool). The behavior of the cars is modelled by two previously typed-in velocity functions in the table in the lower right corner of the interface. Students may insert other velocity functions themselves in the table. The velocity functions that may be inserted in Sim2Bil are polynomial functions, up to third degree[1]. There are also shown two curves (bottom left corner), one for each of the functions, and a shaded region beneath each of the curves. The written formulations of the four task situations are shown in the upper-right corner of Figure 1, while Table 2 presents these task formulations together with potential answers.

Figure 1. Screenshot and description of the basic functionalities of Sim2Bil[2]

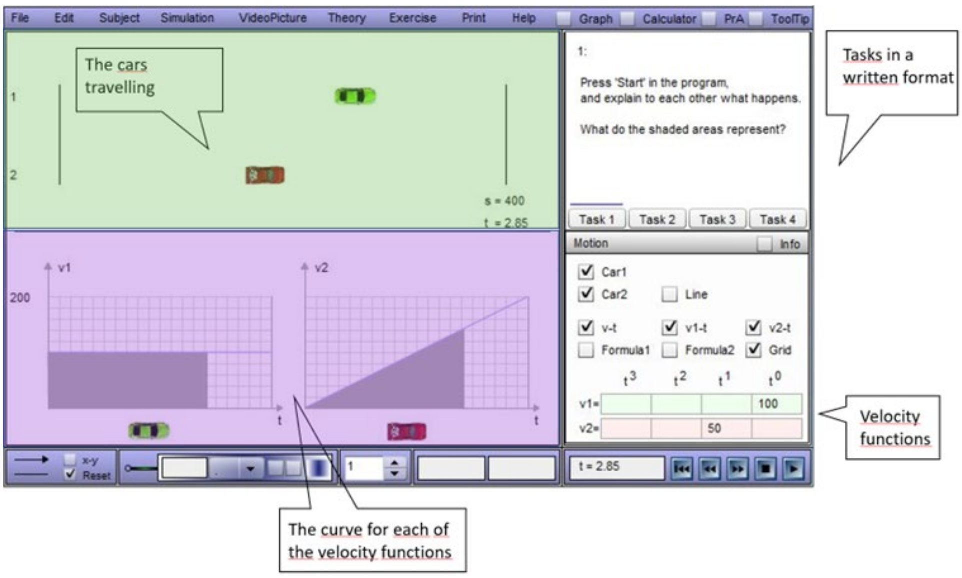


Table 2. Tasks and potential answers to afford task situations

Task number	Task formulation	Potential answers
T1	Press 'Start' in the program and explain to each other what happens. What do the shaded regions represent?	The shaded regions represent distance covered for each of the cars.
T2	Determine other values in the table so that the cars drive with different velocities and arrive at the finish line simultaneously.	$v_1(t) = 6{,}25t^3$ $v_2(t) = 18{,}75t^2$
T3	What can you do to make the green car only halfway when the red car reaches the finish line?	$v_1(t) = 3{,}125t^3$ $v_2(t) = 18{,}75t^2$
T4	Find the velocity to the green car and the red car so that v2 is half of v1 when they arrive at the finish line simultaneously at four seconds. Can you prove that your answer is correct?	$v_1(t) = 100$ $v_2(t) = -25t + 150$

[1] A newer version of Sim2Bil (in which a general functional expression may be inserted) may be found at this link: https://grimstad.uia.no/perhh/phh/MatRIC/SimReal/no/SimRealP/AA_sim/AB/SimReal_Physics_k02_Car_Velocity.htm

[2] Sim2Bil was not designed within a commognitive framework, rather the words used in the tool are to be interpreted in a more everyday language.

The task situations were designed according to general principles inspired by the work of Geiger (2017), Sullivan (2011) and (Oliver, 2000). The situations meet students' prior knowledge of mathematical objects involved and may be approached using different solution paths. There exist several possible answers and students are allowed to make their own decisions and raise their own questions during the process. Within the task situations, we created a facility for social interaction for students to discuss and reason together while being physically together in the same room.

We anticipated that the response by the students would be polynomial functions where the students determine the degree of the functions followed by area reasoning (e.g., the areas beneath the two curves will necessarily have to be equal for the cars to travel the same distance in T2) or integration procedures. It is to be noted that there are several functions that may model the behavior of the cars in accordance with the requirements for each of the task situations.

The students participated in groups at their campus outside of regular lectures. The first author was present during the group session in case the students had any practical or technical questions[3].

3.1.1 Participants

The participants were undergraduate engineering students in Mechatronics at a public University in Norway. In their study program, prior to their participation in the activity, the students had taken an elementary calculus course and were currently enrolled in a course in elementary physics, where the invitation to participate in this study was provided. First year students were invited to participate since many aspects of calculus are explored during the first year, such as areas under curves, the Fundamental Theorem of calculus, as well as velocity and position applications. At the university, first year students from different engineering programs are enrolled in the same course. It turned out that three first year Mechatronic students volunteered, pseudonymized here as Erik, Sam and Tom. Our purpose is not to generalize results across the great population of engineering students, rather to provide "proof of existence" regarding how the interplay between different types of routines may play out. Within the commognitive perspective, routines, while originating in individual engagement with particular task situations, also belong to the mathematical community. Thus, studying this group's mathematical practice reveals mathematical routines belonging to the mathematical community.

[3] It turned out that they had only one question, which concerned how they could verify the travelled distance in Sim2Bil of one of the cars when the other car reached the finish line, cf. T4).

3.1.2 Data collection and data analysis

Our data comprise video recordings (of the computer screen and the students' talk, gestures, and writings) and transcripts from the 45-minutes work session with the task situations 1-4. The transcripts contain the names of speakers, spoken words, and the actions made. Informed consent was obtained from all research participants, their anonymization was ensured by pseudonyms, and ethical approval was given by the Norwegian Agency for Shared Services in Education and Research. We identified the three stages of the identified routines in the transcripts; initiation (init), procedure (proc) and closure (clos). We chose this corpus of data as it allowed us to look for patterns within the students' discourse and whether these patterns changed across the task situations, e.g., to what extent their discourse was deed-oriented.

To identify students' routines, we firstly identified initiations of tasks raised by the students. Secondly, we identified the phases of working with procedures on executing the tasks and the closing of the tasks. The routine stages 'Initiation' and 'Closure' concern the *when* of a routine, while 'Procedure' concerns the *how* of a routine. We characterized a routine as ritualized, explorative or deed-oriented depending on the answers to the questions in Table 3.

A routine is characterized as a ritual when the student is concerned with how to proceed (or which steps they should follow), applies a rigid procedure which they have seen performed by others and the performance is shown to strictly adhere to the rules defining the procedure. The procedure is usually personalized, with the student using words such as "I did…", "You do…" etc. The students do not close their routine by themselves, but expect a final answer which more knowledgeable others need to determine.

A routine is characterized as an exploration when the student is concerned with producing a mathematical narrative (telling stories about mathematical objects), may choose between alternative and applicable procedures (or simply performs the task in more than one way), and closes the routine himself/herself by stating the narrative. In the procedure, steps are bonded to each other (each step is used as an input for the next step). The discourse is objectified, with the student shifting from talking about processes to talking about nouns (e.g., "the definite integral of a function is a constant"). The procedure to be followed is derived from the properties of the mathematical object rather than the concrete object, and the student might exclude herself from the discourse. A certain procedure is enacted for a range of tasks or various procedures might be enacted to solve a given task (e.g., for solving the task "What is the area?" the integration procedure or geometric formulas for calculating areas can be used.).

A routine is characterized as a deed when the student is concerned with affecting change in a primary or discursive object, and the student applies a procedure containing a practical action. By looking into the definitions of routine types, a shift indicates a strengthening in one of these characteristics. The analytical tool (Table 3) is based on the work by Lavie et al. (2019).

Table 3. Analytical tool

	Routine types			
Routine stages		**Rituals**	**Explorations**	**Deeds**
1. Initiation	What are the questions raised by the student?	How do I proceed? (What steps should I follow?)	What do I want to achieve? (mathematical narrative)	What do I want to achieve? (change in environment)
2. Procedure	How is the procedure of the routine determined?	The student applies a rigid procedure that was previously performed by others in similar situations (e.g., replicating integration rules). The procedure is deterministic or nearly so. Personalized discourse (e.g., 'I did…', 'You do…')	The student chooses between alternative, applicable procedures, or performs the task in more than one way (e.g., finding the area under a curve by integration and geometrical reasoning). The procedure is derived from the properties of the mathematical objects (objectified discourse) Each step in the student's procedure is used as an input for the next step (cumulative)	The student applies an unrestricted procedure: may be one of bricolage as long as it brings the required closure; possibly direct – no discursive mediation (such as typing in random numbers in Sim2Bil).
3. Closure	How does the student close their routine?	The performance is shown to strictly adhere to the rules defining the procedure.	Stating the new narrative.	Stating the expected outcome which is judged as adequate; no need for human mediation of the acceptance – it depends on the environment.
	What type of answer does the student try to achieve?	A final answer.	An explicable narrative produced through the performance.	The result – the change in environment – (a change in discursive or primary objects).
	Who determines the end conditions (to indicate the task has ended)?	Others.	Oneself and others.	Sim2Bil and oneself (in the case of no mediating discourse).

The first author transcribed the data and conducted the initial analysis of identifying routines in the transcripts. Thereafter, the second author read through the corpus of data and the initial analysis. The analysis was discussed and finalized. In what follows, we study the interplay between deed-oriented and explorative routines, to reveal possibilities for deeds to enhance explorations as well as for explorations to enhance deeds as the students' discourse becomes increasingly deritualized. The session under scrutiny consists of three collaborating students involved in different routines.

4 Results

In this section, we first give a descriptive overview of the whole session in order to prepare the reader for the forthcoming analysis. Thereafter, we present a more detailed analysis addressing the research questions.

4.1 Descriptive summary of the session

The students familiarized themselves with the first task situation (as presented in Table 2). The students talked briefly about what they saw in the interface of Sim2Bil (as can be seen in Figure 1). Erik started focusing on the behavior of the cars and typed in numbers in the interface of Sim2Bil to change how the cars behaved. When this action did not make the cars meet the requirement in T2 (see Table 2), he shifted his focus towards producing an equation. For Tom, on the other hand, this task immediately made him reason about velocity functions and their appearance. In the beginning of the session, he suggested that they could choose a second-degree polynomial expression by reducing the number of unknowns in a general polynomial expression. He explained different possible solutions to the problem at hand, and that the request was to find one of these solutions. Further on, integration routines and a few arithmetic computations were required. Tom initiated a procedure of integration where he stated that by integrating the expression for the velocity of the cars, they could set it equal to the distance that the cars would run, acknowledging that the integration of velocity is the position. Sam and Tom continued by stating that the areas beneath the two graphs must be equal if the cars cover the same distance. Erik replicated his previous action of typing in numbers in the interface of Sim2Bil, labelled by the students as 'Playing around'. In the following paragraphs, we will show how the students interpret the tasks differently, and how Tom and Sam's more explorative engagement relates to Erik's performance of deeds. The following extracts comprise about thirty minutes of the forty-five minutes the students worked on the tasks.

4.2 Ritualized engagement – trial and error

In the beginning of the session, Erik initiates (in turn no. 31) a procedure of typing in numbers in Sim2Bil (executed in turn no. 40) to find out whether they would make the behavior of the cars fulfil the requirement. He closes the routine (in turn no. 43) when the cars do not behave as expected.

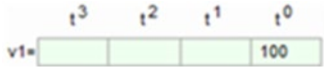
Extract 1. Erik's first engagement with the animation

Turn no.	Person	What is said	What is done
31	Erik	What happens if...	Types in a number in the table on screen
Sam and Erik talk about what they see on the interface of Sim2Bil.			
40	Erik	Let's see how this goes.	Presses the start-button again and watch the cars run towards the finish line
Sam and Erik continue to talk about what they see on the interface of Sim2Bil.			
43	Erik	I actually wanted it [the animation] to go...	The red car arrives approx. half way when the green car arrives at the finish line

Erik's brief utterances, such as "What happens if...?" combined with his immediate actions in Sim2Bil, indicate that he explores the situation to build a conjecture, an important routine in mathematics. Erik's manner of engagement with this routine, however, is highly ritualized: he follows a rigid, trial-and-error procedure, familiarizing himself with the tool while adjusting inputs more or less at random, without drawing on a mathematical narrative. Through these steps, he examines how changes in the velocity function affect the behavior of the cars. He closes his procedure when he sees that the cars do not run as expected (turn no. 43). At the same time, his ritualized engagement involves an element of a deed. Erik's procedure - typing in numbers and running the animation - produces visible changes in objects (the animated cars) directly. We hereby characterize his engagement at the beginning of the session, although mainly ritualized, as also involving an element of a deed. In later incidents, when Erik does not know how to proceed, Sam suggests that they could 'play around' with the animation.

In Extract 2, Tom initiates a new procedure (in turn no. 68) allowing a selection of parameters in a polynomial expression, reducing the number of terms and integrating. Following this excerpt, the students execute this procedure and set the expression equal to the distance of 400 meters. Once they derive two velocity functions, with Tom taking the leading role, Erik starts the animation. They close the procedure when they see that the cars behave as expected according to the requirements.

Extract 2. Tom’s suggestion of reducing unknowns

Turn no.	Person	What is said	What is done
68	Tom	If we cut down here, and take away terms… right? And then we integrate, and then you set it [the distance] equal to four hundred. And then you will still get… Then… then you can make those…extend with those terms we do not want there if you want, right? Okey, we can use t in third and t in first, and then you just let… just set up the equation for those two [cars]. And then we integrate and set equals to as far as we want it to drive.	Points at the table on screen  Points at the empty cell “v1, t3” and “v1,t”.
Tom continues by writing down the expressions. Sam proposes continuing to find velocity functions that have a certain area under its curve. Tom continues with the calculations.			
115	Erik	That must be adequate.	
116	Sam	Yes, I believe so.	

Tom chooses between different alternatives (number of terms) in which all of them will be applicable to perform the task. For answering his own sub-task, which is ‘What degree of polynomial functions do we want to have’, his (sub-)procedure contains elements of a deed (a direct choice of parameters focusing on the movement of the cars – “…as far as we want it to drive”). During the whole session, Tom sees different solution paths, each equally valid, and overall, his procedures consist of cumulative steps – where each step is used as an input for the next. We thus interpret that his overall procedure is explorative. In Extract 2, his word-use changes. Here he explains carefully the process to the others in turn no. 68. He provides the others with a step-by-step procedure while using expressions such as “we integrate…”, “you set…” and “you will get…”. These expressions might indicate a less objectified discourse; however, the procedure is not rigid. For instance, Tom uses expressions such as “we can…”, in which he chooses between applicable procedures. This can be interpreted as Tom adopting a more ritualized way of acting, to invite the others into his discourse. We also find another initiation involving a deed, where Sam suggests finding functions that have a certain area under its curve.

At the end of the excerpt, Erik again turns to Sim2Bil to verify whether their functions meet the requirement. He ends the procedure by concluding “That must be adequate”, which Sam confirms once the animation behaved as anticipated. Although the functions used were derived through Tom’s bonded calculation steps, the substantiation relies on the movement of the cars, that is, drawing on the deed-oriented aspects of the task. Although the procedure employed is mathematical, the substantiation remains tied to the behavior of the cars. We therefore interpret the routine as involving a deed-oriented element.

4.3 Deeds and explorations happening in parallel

We will now highlight situations where deed-oriented and explorative participation happens in parallel. In Extract 3, the students start working on the third question (T3). Tom raises the question whether they can set up a set of equations to find the velocity functions (turn no. 125). Erik, on the other hand, proposes that they "continue to play a little", returns to Sim2Bil and inputs functions there.

Extract 3. Students' negotiation of approaches

Turn no.	Person	What is said	What is done
125	Tom	Can't we just set up an equation set? Eh...let us say that s1 should be half of s2. And t equals two seconds?	
Sam and Erik confirm Tom's suggestion by saying yes and repeating what Tom said.			
130	Erik	Just continue to play a little, right?	
Erik inputs functions into Sim2Bil and dismisses them based on the appearance of the curve. The students continue by returning to Tom's initiation and proceed with integration procedures and other calculations. They start discussing which car should have which expression. Here, Tom and Sam agree that it does not matter which of the cars is connected to which written expression.			
158	Erik	Yes, that thing there?	
159	Tom	Yes, or set the other one equals...	
160	Sam	Yes, or if you set the other one equals two. That doesn't matter. It will be exactly the same.	
161	Erik	Is it okay to see if we get an asymptote? Try that?	
Tom continues his calculations.			
167	Erik	The green one. It is the green one that should be half way and...	
168	Tom	Yes, the green one. Then we must figure out the last things. It was that one that was previously eighteen, so you must exchange that.	Points at the table on screen
169	Erik	Yes, yes that was true.	
170	Tom	This is just half of that here. Agree?	Points towards his written expressions
Once they are done with the calculations, Erik asks for a verification within the tool and Tom writes in their velocity functions in the interface of Sim2Bil: $v1 = 3.125t^3$ $v2 = 18.75t^2$			

175	Erik	It does look like halfway, but can we measure it somehow?	Looks at the green car being in the middle of the interface and the red car at the finish line.
Sam suggests repeating the integration and Erik tries to find the answer in Sim2Bil.			
179	Tom	Well, we have calculated it.	

Tom initiates a procedure for setting up an equation set (turn no. 125). Erik interrupts Tom's initiation by performing a similar routine as in Extract 1, mainly ritualized but including the deed of directly inserting numbers in Sim2Bil, which is closed by a dismissal once the outcome on screen is judged as inadequate. The group then returns to Tom's suggested procedure. Each of Tom's steps is used as an input for the next step in the procedure, and thus the steps are bonded. Sam's comment in turn no. 160, pointing out that it doesn't matter which of the two expressions plays what role, indicates that he is engaging in a similar explorative routine as Tom. Erik, on the other hand, still focuses more on how to produce certain objects – this time an asymptote (the point of which is unclear in this context). Moreover, the closure of their routines differs. Erik searches for verification through the tool (e.g., in turn no. 175), Sam suggests repeating the integration procedures and Tom states that they had done the calculations (turn no. 179). In this way, by looking at what questions the students raise and try to answer, how they engage in different procedures and how they differ in their closure of the routine, we conclude that they are engaged in different types of routines: Erik's engagement is more ritualized and deed-oriented while Tom's and Sam's performance is more explorative.

After several attempts by Tom to shift the groups' course of action towards more explorative activity, Erik's engagement indeed shifts towards becoming more explorative. We will exemplify this shift in Extract 4 which takes place after the group have read T4 (see Table 2).

Extract 4. Shift towards more explorative engagement

Turn no.	Person	What is said	What is done
225	Sam	Oioioi. That was a complicated one. Ehm...	
226	Erik	We can in fact make it a bit simpler. And just say that the green car drives with constant speed. That might be easier to find?	
227	Tom	That... that is smart to say that this one [car] should drive with constant velocity.	
Erik sets v1=100. The students agree to find a certain function that has the numerical value 50 when t equals four.			
While Tom spends some time on the calculations (and says what he is doing while writing), Erik inserts v1=100 and v2=50t in the interface and plays the animation and dismisses the result in the following utterance.			
263	Erik	Mhm.... We must in fact do the calculations.	

Tom starts to explain the calculations and Erik comments on what he sees in the interface and says that they can do calculations to solve the problem. After Sam has produced two functions, Erik types them in the interface. Without playing the animation, he dismisses the result by the following utterance.			
288	Erik	I tried to take minus fifty there [pointing at the interface] and then it will go like that. So, then they will reach at the same time, but it will stop there. It shall be half the speed when it stops there. So it must have a...	Makes a gesture of a parabola
Erik continues to explain his reasoning to Sam.			
297	Tom	Yes, look here. Ehm... Now, we have written the integrals but.... Okey. We know that when v2 is on four seconds, it should be half of v1, right?	Pointing at their writings

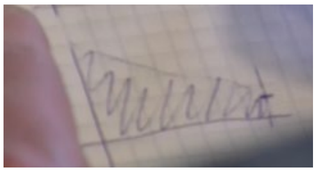
In the above excerpt, the students' focus is on the deed of making the cars behave in a certain way (turn no. 226). Sam's perception of the task as complicated, prompts the group to simplify their procedures by setting $v_1 = 100$. Erik persists in deed-oriented activity by inserting numbers into Sim2Bil to see what will happen, playing the animation and quickly dismissing the result as unsatisfactory. Finally, in turn no. 263, he acknowledges that they must mathematize the problem through calculations in order to produce the velocity function, that is, producing a narrative rather than a certain discursive object (functions, curves or areas) or a transformation of primary objects (the cars). In focusing on formulating narratives about producing functions rather than on how to get particular functions, Erik is shifting towards a more explorative engagement. Still, his focus is on the doing ("we must in fact do the calculations"), suggesting elements of ritual still remains.

To find the other function, Tom proceeds with calculations and integration procedures. While Tom does this, Erik changes his focus back to 'playing around' with the animation. Erik dismisses Sam's functions by a deed-oriented argument about the cars' behavior (288). When Erik explains how the cars need to travel to meet the requirement, his reasoning is based on their work in the previous task situations and seeing the animation play out with several velocity functions inserted in the interface of Sim2Bil. The way that he supports his mathematical arguments by referring to the behavior of the cars in Sim2Bil suggests that his exploratory engagement is supported by his previous deed-oriented activity. Once Erik dismisses the functions in turn no. 288, there is a turning point where Tom uses the opportunity to bring their focus back to producing a narrative (turn no. 297).

4.4 Discursive shifts between explorations and deeds

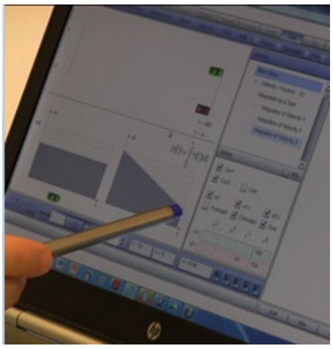
We will continue to focus on the interplay between deeds and explorations. In Extract 5, Tom suggests they can set up a matrix. This extract is another example of Tom's attempts at shifting Erik's engagement towards becoming more explorative.

Extract 5. Discursive move by the more well-versed interlocutor towards more explorative engagement

Turn no.	Person	What is said	What is done
352-354	Tom	Those two different expressions. That is those two, them... This one says that when it is at the finish line, that is, after four seconds it has gone four hundred meters.	Points at his writings $2C_1 + 4D_1 = 400$
		And this one says that after four seconds, the velocity should be fifty.	Points at his writings $4C_1 + D_1 = 50$
		Yes, right. Yes, so no matter what... set up a matrix or what you want then right... you will get two, four, four hundred, four, one, fifty, right? So...yes. It is just to use the insertion method, right? Take the matrix... like... but then...	
Erik explains to Sam how he sees what the animation should be like.			
363-369	Tom	No, But I think you.... Look here. If you think like he said, areas, right?	Points at Sam
		So, you have this one. Ehm...it is... it should have constant velocity, right? It is like this here. Right?	Makes two drawings and starts to shade the first one: 
		(...) And then you have the other one. That should just, that should end on half of here, right? So, it should hit here. And then it should begin up here, and it shall have the same area like this one.	
		Ehm... So, I mean that you can just draw a line from the middle there then. So the area here.... You can find out how high up it must start. And the slope on this one.... because it should...yeah. You just need the same area on this one if you think trigonometrical.	
The students discuss the calculations.			
424	Tom	Hundred times four. It should be the same as fifty times four plus... and then you will get a triangle. Eh... the areas of a triangle where you know...	

In Extract 5, Tom first (352-354) clarifies his procedure by speaking about the behavior of the physical objects. His explorative routine use is again expressed as a step-by-step procedure the others can follow – initiating a ritualized way of engaging with the task. Later, he draws two areas to explain that they can produce functions that will make the two areas equal in size (363-369). Thus, across Tom's attempts to invite Erik into his discourse, he first uses a ritualized approach, giving a pure "recipe" with steps of how they should proceed, but then shifts towards also bringing in elements of a deed by realizing the mathematical expressions as objects and including the result of the narrative (e.g., what the two areas should look like). To invite the others into his discourse, his initial explorative engagement, focusing purely on producing functions (see e.g., Extract 2, turn no. 68), shifts towards getting a certain result (e.g., areas). In the above extract, when he explains the steps and draws the two areas, he performs what Sfard and Lavie (2005) term "interwoven deeds" – practical actions (drawing, shading, pointing) aimed at producing certain areas - that are closely intertwined with his mathematical narrative and served as part of the mathematical argument itself. In Extract 6, the students discuss how they can substantiate their answer to the last task (T4).

Extract 6. Making the substantiation explicit – The closure

Turn no.	Person	What is said	What is done
470	Erik	How can we prove it then? It is what is asked there.	
471	Tom	We have indeed proven it. We have proven it with those there.	Taps his pen on his writings.
The students agree that they are done with the calculations.			
488-490	Erik	Well, we can also see it here that it stops on half of the other. So, it stops at forty. So, you can actually see it on the curve as well.	Points at the second curve on the interface. 
491	Sam	Oh yes. And you also see that there is a connection between the area under the curve which is the integrated of the velocity....	
Sam and Erik talk about the interface.			
499	Tom	But it is in fact enough with these here.	Points at his writings.

The students' discussion of how to substantiate their discursive actions reveals that the origin of substantiation differs between the students: Erik uses the curves in Sim2Bil to argue for the substantiation (488-490), while Sam uses the connection between the area under the curves and the integrated velocity to explain why their narrative holds (491). Lastly, Tom states that the substantiation lays within the mathematical narrative itself (499). Where for Tom the substantiation arises from the discourse itself, Erik seeks a substantiation in the digital tool.

4.5 Towards exploratory engagement

We argue that there is a change in Erik's engagement within the task situations. In the first task situation, he performs rituals of a deed-oriented nature (a direct transformation of the cars). By the deed of choosing functions to transform the behavior of the cars in Sim2Bil, he realizes that there is no direct way to tackle the problem at hand without a prior exploration of areas, curves, or polynomial functional expressions. These types of explorations may be performed independently of deed-oriented engagement aimed at changing the behavior of the moving cars, as the students may investigate such mathematical objects without entering functional expressions into Sim2Bil. In fact, such initial explorations might be the only way to tackle the problem at hand. Indeed, even though the tool is accessible and preferable for Erik, his functions are not directly translatable into a transformation of the movement of the cars to accomplish the tasks. An exploration of the velocity functions is needed to proceed beyond simple trial-and-error attempts.

In all the task situations, Tom leads the discourse towards an explorative engagement. However, Erik corrects what the others are saying when it is not in correspondence with the already agreed result. Their shared outcome serves as a basis for how to proceed and allows them to adjust their procedure along the way when the path is not leading them to the expected result, and to dismiss less fruitful operations before even executing them. Over time, Erik's engagement shifts as he accepts and follows Tom's more explorative approach.

Tom's ritualized attempts to support the individualization of Erik's discourse, even though Erik already understands the purpose of the new routine, are not unique. Most attempts at individualizing others' discourse result in rituals rather than explorations (Sfard, 2008). It's likely that the discussants may become familiar with the procedure but not when to apply it (Sfard, 2008). In line with this, Erik follows Tom's steps when prompted, but struggles to do so independently. When uncertain, he initially treats the task like a guessing game by talking about or performing the deed of randomly choosing functions to run the cars. As the deed of choosing is integrated with other procedures, the engagement transforms into a choice-enhancing exploration.

Over time, the performance of deeds helps establish a shared understanding of the final result in a physical context and allows Erik to confirm or challenge the narratives formulated by the other students. In the final task situation, Erik's discourse changes as he independently begins formulating a mathematical narrative based on feedback from Sim2Bil and on Tom's attempts at facilitating a common explorative engagement. Throughout the session, deeds and explorations are intertwined, influencing the students' actions. This mutual dependence between the two is particularly evident in the case of Erik.

In the above analysis, we have attempted to figure out the inner logic of the discourse comprising both rituals and explorations and some elements of deeds. The two initial procedures used by Erik and Tom (see Extract 1 and Extract 2), directly choosing functions and integrating functions, are performed independently. Over time, they evolve into an integrated procedure, where the focus is to arrive at the same result (that is: the cars fulfilling the task requirements). Figure 2 shows Tom's two ways of inviting Erik to shift his ritualized routine use towards something more explorative. First, Tom presents his explorative routine as a ritual, to clarify how the procedure should be executed. Thereafter, he brings in elements of a deed to clarify the result of the procedure. In Figure 2, the two invitations to explorative routine use are illustrated: either through ritualized routine use or by bringing in elements of deeds towards more explorative routine use.

Figure 2. Two ways of invitation to explorative routine use

5 Discussion

This study investigates how a group of engineering students navigate between ritualized, deed-oriented and explorative routines when working with a dynamic animation tool, and especially how the deed-oriented and explorative routines interact in the students' mathematical discourse. Across the four task situations, the students' engagement revealed a complex and iterative relation between deed-oriented and explorative routine use. In this section, we synthesize our findings, discuss their implications for commognitive research and outline limitations of the study.

5.1 The interplay between ritualized, deed-oriented and explorative routines

A main finding of the study is that deed-oriented and explorative routines did not stand in contrast to each other, but rather informed and shaped one another. The students' engagement exemplifies how deed-oriented routines, such as entering functions in Sim2Bil to manipulate the travel of the cars or producing certain areas, provided a common ground for the students. Erik's engagement in particular exemplifies how deed-oriented routines supported his later attempts to formulate and endorse mathematical narratives. Our analysis suggests that this process is characterized by intricate interplay, with students moving back and forth between the two types of routines.

Erik's repeated shifting between ritualized and explorative routines, resonates with the notion of *the de-ritualization process* described by e.g. Lavie and Sfard (2019) and Sfard and Lavie (2005). Erik's engagement, as described in this article, aligns with previous research: there is no simple transfer between deeds and explorations (Sfard, 2005; Sfard & Lavie, 2005). The observation extends this research by showing how deed-oriented routines may support more explorative routines. In Erik's case, deed-oriented routines allowed him to challenge and verify the students' mathematical narratives. This suggests that deed-oriented routines may act as meaningful components of more explorative engagement. It is important to acknowledge that invitations to other routines are interpreted and perceived differently by students, as concluded in Wood (2016).

Historically, learning mathematics was thought to stem from familiar and everyday reality, but this process is complex and not always effective (Sfard, 2005). Sfard (2005) shows how discourse is not easily detached from deeds (manipulation of objects such as coins or numbers). In our case, the intermediate result was achieved by playing the animation. In Sfard (2005) the computational discourse was needed to perform the numerical calculations before passing the correct amount of money.

The mathematization that occurs moves the discourse horizontally (to precede the practical actions as manipulation of coins, and in our case is the manipulation of cars) and vertically (to a discursive meta-level where one operates on discursive constructs, in our case functions). This "zigzagging movement" (p. 74), in which one builds upon the other, is a demanding task (Sfard, 2005) which Erik partially succeeds in. Separating discourse from transformations of primary objects requires substituting them with symbols and being explicit about the arithmetical operations and the intermediate results obtained (Sfard, 2005). Once Erik mastered the deed, he was less inclined to renounce his procedure to try new, more applicable procedures, which aligns with earlier findings which indicate that appreciation for such new procedures often comes only after use (Sfard, 2005).

The tool and the task instructions contain a hidden discursive 'scaffolding', a past discourse (Sfard, 2005), i.e., the tool and tasks have been designed for educational purposes based on prior knowledge about the mathematical objects involved. Since this discourse is embedded in Sim2Bil, it's not surprising that students needed time to familiarize themselves with its functionalities. Our students' familiarization involved transforming the moving cars to meet the task requirements. We suspect that when using a new tool, students' routines may start as deeds. Also, the transformation of moving cars in Sim2Bil was highly accessible to the students. If the velocity functions involved were less easily translatable into a direct transformation of moving cars, their choices of procedures might have been different. The students might instead have started to mathematize the problem.

5.2 Revisiting Sfard's framework in the context of engineering education

Our findings also contribute to the theoretical perception of deed-oriented and explorative routines by situating them within a specific context; engineering education. Prior research on deeds (e.g. Lavie et al., 2019; Sfard, 2005) has mainly examined children's engagements with primary objects. As far as we have been able to determine, deeds have been perceived more or less in the same way since Sfard's introduction of the term in 2005. Classic examples, such as money transactions (Sfard, 2005, 2008) and numerical calculations (Lavie et al., 2019), illustrate what deeds look like in practice. While mastering deeds is not the primary goal, our study concludes that having a goal that changes objects can inform the discursive activity and support the production of a mathematical narrative. In our students' discursive development, deeds played a different role than in the cases of children's learning reported in the literature (e.g., Lavie & Sfard, 2019; Sfard, 2005). We hypothesize that this is connected to the particularities of engineering education, where mathematics functions as a service subject rather than something to be learnt in its own right. In fact, much use of mathematics in engineering contexts will take the form of deeds, where the goal of performing a mathematical procedure will not be producing a new mathematical narrative, but rather producing a numerical result which is then used in solving some real-world engineering problem. Thus, we argue that the interplay between deeds and explorations may be beneficial for learning, at least in such contexts.

We assume that fluency in engineering discourse, as in any discourse, involves the ability to alternate between different types of routines. For making a transition from ritualized to explorative engagement, one needs to think about the performed operations as a 'storytelling activity' (Sfard, 2008, p. 240). The students need to see integrations as not only resulting in new objects, such as velocity functions, but also as producing new endorsed narratives about these integrations and functions. The mechanism behind such a discursive change lies, in general, in the modification of rules defining patterns in their activity (metarules) and in the introduction of new mathematical objects mediated by an experienced interlocutor (Sfard, 2008). For Erik to transform his routine of choosing functions, one of the other students needed to drive the discourse forward along a different path. Tom's more explorative engagement facilitated a substantial change in Erik's discourse by transforming the routine of choosing into an exploration through a ritualized way of acting.

While deeds have been said to represent practical learning with limited scope (e.g., Sfard, 2005), explorations enhance them by making procedures more universal and independent of situated visual mediators (such as the cars, areas and curves in Sim2Bil) existing outside of the discourse (Sfard, 2005). When finding a direct answer using Sim2Bil failed, a more explorative routine for producing velocity functions emerged. This shift is crucial for engaging in mathematical discourse and addressing new problems. Our data also show how deeds can enhance explorations, when a deed-oriented routine was incorporated to clarify and modify explorations. Indeed, the expected outcome of the cars' behavior was used as the students' common goal and modified their mathematical narratives when the cars didn't behave as expected. As documented engineering activities show, deeds may inform the students' explorations to adjust their models to become more functional in real life contexts (c.f. Gainsburg, 2006). As Stillman et al. (2020) say: "… models might not provide desirable outcomes so the whole process is iterative allowing for both refinement and further extended exploration of the beginning situation" (p. 1215). This iterative process was exemplified when Erik modified functions based on the observed behavior of the cars. His transformation of this behavior gave him an object that the group could all agree upon as the end result that they were seeking, and also as a departure point for how to get there. Even though the final outcome of the movement of the cars was clear to the students the path of how to get there was not straightforward for them.

Sfard (2008) distinguishes between explorations and deeds as two different types of activities. Our study shows that, despite their differences, the two activities may inform each other. In engineering students' work, both refer to inseparable, but different, facets of the same coin. We thus propose that deeds and explorations be contextually seen as a dyad rather than a dichotomy. In the latter, the two are viewed as different and distinct routines that are not compatible or co-existing. Rather, we claim, in the learning process deeds may enable the development of explorations. In this way, the deed should be acknowledged for its role in discursive activities. The constructs ‘deed’ and ‘exploration’ might turn out to correspond to what Hiebert and Lefevre (1986) observed with reference to the issue of ‘concept’ and ‘procedure’: “Historically, the two kinds of knowledge have been viewed as separate entities, (…) at best coexisting as disjoint neighbors. (…). In contrast, there is a growing interest today in how concepts and procedures are related” (p. 2). We argue that this fits well with the work by Lavie et al. (2019), where the ritual-exploration dichotomy was dismissed in favor of a more fluid view where they are seen as endpoints of a spectrum. Since Lavie et al. (2019) group deeds with explorations as product-oriented, contrasted with rituals which are process-oriented, this indicates that the spectrum from ritual to exploration might just as well be seen as stretching from ritual to deed, and that the distinction will be determined by what the final goal of the activity is, something that might be subject to change. We hypothesize that our analytical result is not an outstanding incident but that there are other examples out there that need further substantial investigation.

That mathematics in STEM subjects, e.g., engineering, serves as a tool for solving real-world problems makes it crucial to study the role of deeds in engineering education and how it might differ from the role deeds have in children’s performances. University-level mathematics in the context of engineering education differs from school mathematics in that it is to a large extent used to transform physical objects rather than telling stories about mathematical objects. The objects operated upon are often primary, material objects, e.g., a tower, a machine, or a robot. While mathematical narratives remain important, the focus shifts to transforming material objects. In the first-year studies, the mathematical objects are taught with less focus on their applications (Loch & Lamborn, 2016). However, later courses integrate mathematics fully into engineering knowledge, making them inseparable so that the latter is in practice as “inseparable from mathematics as music is from sound” (Schmidt & Winsløw, 2021, p. 282). This inseparability of mathematical objects and their applications complicates the investigation of deeds as distinct from explorations. As Sfard (2008, p. 239) notes, the difference between the two becomes elusive when the objects of deeds are discursive rather than primary. Additionally, the same sequence of mathematical operations might be seen as an exploration for one student and a deed for another (Lavie et al., 2019), with the distinction emerging from how students talk about their actions.

The reported findings must be interpreted in light of the limitations of the study. Firstly, the data are based on a limited number of participants and a limited time frame for the session. This restricts the generalizability of the findings. However, our goal at this stage is not to make general claims that address all engineering students, but rather to examine how certain routines may interact in a particular instance of mathematical discourse. Still, there is nothing particularly unique about the students in this study, nor in the context in which they were working, acknowledging that the use of a digital animation tool may have encouraged more deed-oriented routines than other environments would have. Thus, we see no reason to believe that the kind of interplay between deeds and explorations we have displayed in our data could not be detected in other situations where students engage in mathematical problem solving using digital animation tools, or even in STEM students' engagement with mathematics more generally.

5.3 Implications for teaching and research

This study suggests several implications for future research on students' routine use in mathematical discourse and for designing learning environments in engineering education. Given the observed interplay between deed-oriented and explorative routines, teaching approaches may benefit from acknowledging this interplay rather than seeking to replace deeds with explorations. We suggest carefully incorporating elements of deeds in teaching and learning of mathematics and not discarding them as unfruitful for mathematical learning in engineering education. Deed-oriented routines may serve as a point of departure for more explorative routines, given appropriate scaffolding. Digital tools, such as Sim2Bil, which provide immediate visual feedback, may allow explorations to emerge. Educators may therefore consider designing tasks that facilitate deed-oriented routines as an integral part of the construction of mathematical narratives. Educators might support students' gradual transition between routines by prompting them to articulate their reasoning behind constructed mathematical narratives.

Further research is needed to explore the role of deeds in engineering education, whether their role changes during the study years, and how they contribute to students' mathematization. In this article, we have attempted to pave the way for the latter.

Sfard (2005) suggests that matching deeds with explorations involves seeing how manipulation of symbolic mediators may replace physical transformations of primary objects. However, the ability to alternate between deeds and explorations is essential (Sfard, 2008), especially in engineering education. Our students recognized integrated velocity functions as distance travelled early on, and recalled this relation at the end of the session. Erik's movement between deeds and explorations showed that fully integrated alternation between the two takes time.

Finally, we would like to stress that more research is needed to further study how students may effectively navigate between different routines. This article sheds light upon the interplay and the potential relationship between deeds and explorations. We hope that it may act as a catalyst for further research on deeds and contribute to research on the possibilities for deeds to enhance explorative routine use, and the other way around, in the complex interplay between the two.

Research ethics

Author contributions

Both authors contributed to the study conception. Ninni Marie Hogstad performed the material preparation, data design, data collection and formal analysis. The first draft of the manuscript was written by Ninni Marie Hogstad and Olov Viirman and both authors commented on previous versions of the manuscript. Both authors have read and approved the final manuscript.

Artificial intelligence

Artificial intelligence has not been used in the research or in writing the article.

Funding

This work was supported by the Centre for Research, Innovation and Coordination of Mathematics teaching (MatRIC).

Institutional review board statement

We comply with international, national and institutional standards of research reporting, including informed consent from the participants and the participants are anonymized in the manuscript. The research is approved by the Norwegian Agency for Shared Services in Education and Research (project number 42497).

Informed consent statement

Informed consent was obtained from all research participants.

Data availability statement

The transcriptions of the data associated with the paper can be accessed by request. The video recordings cannot be made open due to privacy protection of the participants.

Acknowledgements

We are very grateful to Professor Martin Carlsen and Professor John Monaghan for helpful comments on earlier drafts of this article. We also wish to thank the students for their participation in the study. We would like to thank also the reviewers who have provided us with valuable insights into how to improve the quality and structure of our article.

Conflicts of interest

The authors declare no conflicts of interest.

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