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Research article

Pre-service teachers using an app to generalize figural patterns through figural reasoning

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Abstract: To investigate a dynamic learning environment's (DLE) facilitation of students' figural generalization of figural patterns, this study scrutinizes 12 pre-service teachers' efforts to generalize figural patterns using a DLE that offers dynamically adapting shapes. The shapes support identification of figural patterns' figural commonalities. In video recorded task-based interviews, the pre-service teachers worked in pairs to solve figural pattern tasks using the DLE. Duval's (2006) theory of semiotic representations is utilized to identify characteristics of the pre-service teachers' conversions and treatments. Results show that they adopted an experimental approach and utilized the DLE to create multiple valid symbolic generalizations, and they used symbolic treatments to support some solutions. However, they would often treat only one exemplar of the figural pattern, weakening the basis for their generalizations, and they struggled to express verbally the figural patterns' generalized structure. These results raise concerns about the algebraic thinking involved in their solution processes. Implications of this study include the identification of crucial factors of DLEs designed to support students' exploration of figural patterns.

Keywords: algebraic thinking, dynamic learning environment, figural patterns, generalizations

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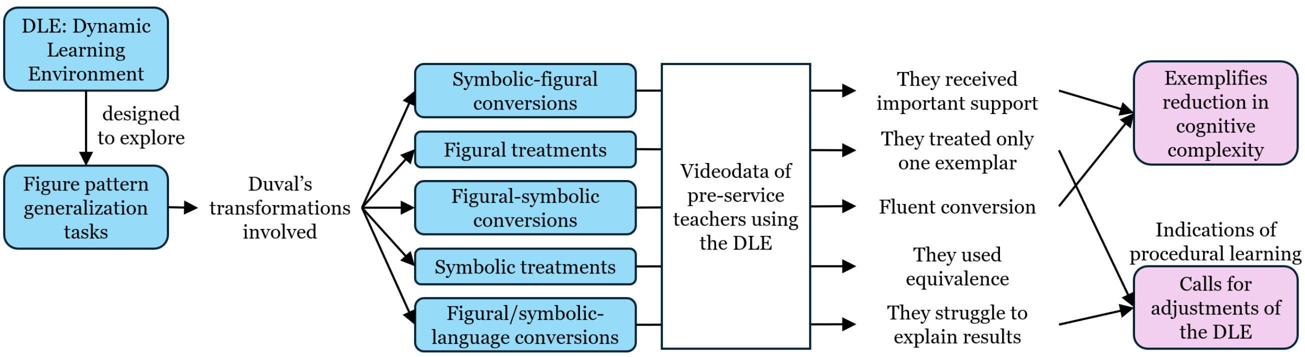
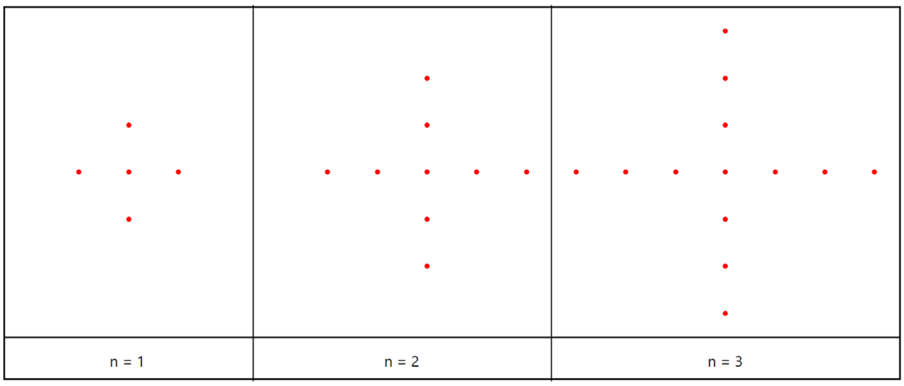


1 Introduction

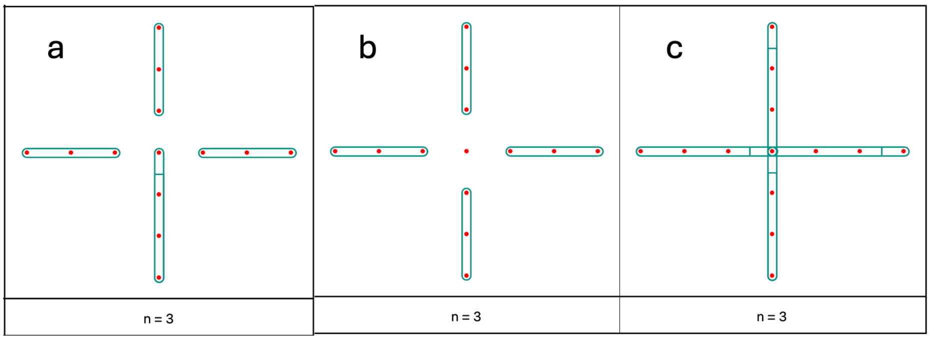
Mason (1996) approached algebraic thinking through the concept of generalization. He claimed that acts of generalizations, characterized by detecting sameness and difference in essential features, are the basis of algebraic thinking. In the effort of teaching students to generalize, figural pattern tasks are often used (Jackson & Stenger, 2024), which Radford (2008) regarded as a “route to algebra” (p. 84). In such tasks, students are typically provided with the first three exemplars of a growing pattern (Figure 1) and asked to draw consecutive exemplars, find the number of elements in exemplars close by or far away, or create a general (symbolic) rule to find the number of elements in any exemplar. Here, we refer to the process of creating such general rules as figural pattern generalization.

Figure 1. The Cross Numbers



In generalizing figural patterns, students most often derive symbolic expressions either numerically through inspection of the number of elements in the visualized figure exemplars, (*numerical generalization*) or figurally through identification of commonalities in the exemplars’ figural features (*figural generalization*, Bills & Rowland, 1999; Dörfler, 1991; El Mouhayar, 2018; Küchemann, 2010). Figural features are the patterns’ geometric, perceptually available terms (Rivera & Becker, 2007), exemplified by the four ‘arms’ in Figure 1. A figural generalization of this pattern may involve identifying the monotonously increasing number of dots in each arm, making the length of each arm similar to the exemplars’ value of n. This is a commonality of the figural pattern exemplars. Thus, we have found “a way of structuring and organizing counting” (Hewitt, 1998, p. 20), illustrated in Figure 2 for $n = 2$. A figural pattern may often be structured in many ways, and two other ways are displayed in Figure 2.

Figure 2. Three different ways of seeing commonalities in The Cross Numbers (in the $n = 3$ exemplar), giving rise to the equivalent symbolic generalizations $(n + 1) + 3n$ (a), $4n + 1$ (b), and $4(n + 1) - 3$ (c). Note that the $n + 1$ stripes are stripes designed with a small bar, included to single out the "extra" dot corresponding to $+1$ in the expression $n + 1$. All four stripes in (c) include the center dot



Mathematics educators argue that approaches to generalizations drawing on figural features fosters algebraic thinking and provides reasons for why a generalization is valid (e.g., Hewitt, 2019; Küchemann, 2010). A figural generalization supports students in articulating the relationship between a figural pattern and a generalized rule (Yao, 2022). However, researchers have documented students' tendency to generalize figural pattern tasks using numerical approaches, for instance by inferring numerical relationships between figural pattern exemplars (e.g., Becker & Rivera, 2005; El Mouhayar, 2018; El Mouhayar & Jurdak, 2015; Healy & Hoyles, 1999; Lannin, 2005). Several researchers provide evidence for the weaknesses of numerical generalizations, where students solve figural pattern tasks in a tabular manner (e.g., Hewitt, 2019; Küchemann, 2010; Montenegro et al., 2018). For instance, Küchemann (2010) argued against the practice of counting the elements in the first exemplars and continue with "no reference to context, that is, the nature of the given (...) pattern" (p. 235). He claimed that this practice distances students from the search for structure.

As the literature review will show, there is a lack of knowledge on how to support students' figural approaches in figural pattern generalization tasks. In particular, the potential support of dynamic learning environments (DLEs) is not explored. Yeo and Webel (2024) describe such dynamic learning environments as digital tools designed to support learning, where "dynamic tools can be discrete (...) or continuous" (p. 21). DLEs open "new possibilities for visual expression in the process of mathematical reasoning" (Healy & Hoyles, 1999, p. 59). DLEs may relate visual and symbolic representations and enable discernment of invariants (Leung et al., 2013), and they may provide immediate feedback (Ruthven, 2018). Dyrvoll and Bergvall (2023) argue that such learning environments foster persistence and exploration. The purpose of our study is to address how a DLE may support students' figural generalization of figural patterns. In this article, we present results from a study of pre-service teachers (PSTs) enrolled in the grade 1 to 7 program using a DLE that facilitates figural approaches. The DLE in focus is the Figural Pattern Computer Application, referred to as the *FPapp*, developed by Author 2. We describe PSTs' generalization of figural patterns using the FPapp and scrutinize the support provided by the FPapp drawing on Duval's (2006) cognitive theory of semiotic representations. Drawing on the results presented in this article, we find reason to critically discuss the PSTs' explanations of their generalizations. Arguably, the FPapp has shortcomings as a DLE. By analyzing not only the positive effects of the FPapp, but also the effects of its shortcomings, we can identify essential properties of DLEs aimed at supporting students' figural pattern generalization.

2 Literature review and research question

2.1 Theoretical framework

Duval's (2006) theory of semiotic representations is a cognitive approach to learning. Cognitive approaches focus on how individuals mentally process and organize mathematical knowledge. Duval's theory has proved itself useful for studies of algebraic thinking (e.g., Bråting & Kilhamn, 2021; Montenegro et al., 2018; Yao, 2022), and it provides terminology well suited for investigations of generalizations of figural patterns.

In Duval's (2006) view, mathematics is a unique area of knowledge due to the total dependence of its objects to be semiotically represented. A representation is "something that stands for something else" (p. 103). Since the representation of a mathematical object cannot be compared directly to the object, the learner must infer which properties of the representation are mathematically relevant. For instance, the number of dots in each exemplar in Figure 1 and the cross-like shapes are relevant properties of 'The Cross Numbers'. However, the absolute distance between the dots is an irrelevant property.

A semiotic representation system consists of a set of representations of mathematical objects (signs) and a set of rules and associations between their signs. The main property of a sign is its capacity to be transformed into other signs. A semiotic representation system that allows transformations, is called a semiotic register. In this article, the figural register refers to the (geometrically) visualized figural pattern. The symbolic register refers to the symbolic algebra, and the language register refers to the verbal use of the natural language.

Different registers facilitate different mathematical processes. For instance, determining the number of dots in the 125th exemplar of The Cross Numbers using the figural register of visualized dots would be a tedious job. Using the symbolic register, however, we can multiply 125 with 4 and add 1, quickly concluding that the 125th exemplar contains 501 dots. Furthermore, different registers make explicit different properties of the same mathematical object. For instance, symbolic descriptions of The Cross Numbers do not convey information about its geometrical structure.

According to Duval (2006), mathematical knowledge is produced through the transformation of signs. Transformations that happen within a register are called *treatments*. Some registers exist mainly for the purpose of mathematical processing, with most treatments being procedural. These are called mono-functional registers and are exemplified by the symbolic register. Other registers have multiple cognitive functions, like imagination, processing, and communication. These are called multifunctional registers, and treatments in these registers can rarely be formulated as procedures. In discussing sources of incomprehension in mathematics, Duval (2006) makes the point that treatments in multifunctional registers are cognitively more complex than treatments in monofunctional ones. Figural registers are multifunctional registers, whose complexity may be exemplified by students' struggle to see figures with an awareness of possible sub-configurations and opportunities for reconfigurations. Figure 2 exemplifies treatments conducted on The Cross Numbers, illustrating three different ways of seeing sub-configurations in a figural pattern.

Conversions are transformations of representations from one register to another register "without changing the objects being denoted" (Duval, 2006, p. 112). For example, the stripe containing the center dot and the three rightmost dots in $n = 3$ in Figure 2, converts from the figural register to $n + 1$ in the symbolic register. Duval argues that conversions are cognitively more complex than treatments. To make a conversion, one must coordinate (at least) two different registers and recognize the same mathematical object represented in two different ways. For instance, a figural generalization includes converting the result of a figural treatment of figural pattern exemplars into symbolic expressions. Considering The Cross Numbers' increasing 'arms' (Figure 2, $n = 2$ exemplar) as mathematical objects, one must recognize how one 'arm' in the figural register may be represented by the letter n in the symbolic register.

2.2 Generalization of figural patterns

Radford and Peirce (2006) articulated an acknowledged description of the cognitive capabilities involved in making an algebraic generalization of a pattern: "(…) grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all the terms of S and being able to use it to provide a direct expression of whatever term of S" (p. 5). According to Radford (2008), solving figural pattern tasks without identifying commonalities does not promote algebraic thinking. He exemplified this with naïve induction, where students guess rules and verify them through checking the number of dots in one or two exemplars. Furthermore, Radford and Peirce (2006) argued that not all symbolization is algebraic. The symbols need to designate indeterminate objects. Algebraic generalization rests on a generalization of a local commonality to all the terms of a sequence, serving "as a warrant to build expressions of elements of the sequence that remain beyond the perceptual field" (Radford, 2010, p. 42).

Contributions in the past 20 years of research on students' efforts to generalize figural patterns include the research of Rivera and Becker (e.g., 2007, 2008) and El Mouhayar and Jurdak (2015, 2016). Rivera and Becker (2008) offered a discussion on the cognitive act of 'grasping', 'noticing', and 'seeing' commonalities. In the context of figural patterns, they drew on Duval's (1998) concepts of perceptual and discursive apprehension, where the former involves seeing a figure as a single gestalt and the latter involves seeing a figure as a configuration of multiple gestalts. Rivera and Becker (2008) distinguished between two ways of generalizing within a discursive apprehension. Taking a constructive approach means treating figural patterns as consisting of non-overlapping parts. This is illustrated in $n = 1$ and $n = 2$ in Figure 2. A deconstructive approach means seeing a figural pattern as being constructed of possibly overlapping sub-configurations, as illustrated in $n = 3$ in Figure 2.

Rivera and Becker (2007) investigated students' use of numerical and figural features in their abductive-inductive approaches to generalize figural patterns. They concluded that those who approached the tasks numerically often failed to provide a conceptual explanation. Some of these students' abductions relied on 'guess-and-check'. On the other hand, those who relied on figural features focused on invariant relational structures and introduced variables consistently. They were more capable of justifying the viability of their generalizations.

Building on the work of Radford (2008) and Rivera and Becker (2007, 2008), El Mouhayar and Jurdak (2015, 2016) and later El Mouhayar (2018) investigated students' generalization approaches. They made a distinction between *numerical* reasoning approaches, where generalizations are inferred from numbers generated from figure exemplars, and *figural* reasoning approaches. The latter is a merger of Rivera and Becker's (2007) 'figural similarity', where generalizations are inferred from figural relationships between figures, and Küchemann (2010) 'structural figural' generalization, where a generic case is analyzed.

Furthermore, El Mouhayar and Jurdak (2015, 2016) documented how generalization approaches differ with respect to grade level and types of generalization tasks. Moreover, they exemplified how different strategies may be applied in numerical and figural reasoning approaches (El Mouhayar & Jurdak, 2016). A recursive strategy in a numerical approach involves identifying a numerical difference between consecutive terms and reaching the next terms by repeatedly adding this value. In a figural approach, a recursive strategy means recognizing the structural growth between consecutive figures and adding this to the given exemplar to reach the next figural step. With respect to our study, where PSTs were asked to create symbolic generalizations, the functional strategy is most relevant. A numerical approach to this strategy means recognizing a rule based on the numeric pattern. A figural approach to a functional strategy involves identifying the constant and the growing components and relating the growing components to the figural step number.

We extend this review by incorporating two publications that applied Duval's (2006) theory to the study of figural pattern generalization. Firstly, Montenegro et al. (2018) classified the students' actions as they generalized figural patterns using the concepts of treatments and conversions. The students' numerical approaches provided limited success. Rather, performing treatments on the original figure, like circling dots as shown in Figure 2, provided students with favorable conditions to generalize. Montenegro et al. (2018) argued that conducting figural treatments on figural patterns supports the identification of commonalities and "opens the way for conversions between representations" (p. 106), which according to Duval (2006) is crucial for mathematical learning.

Secondly, Yao (2022) applied Duval's (2006) theory and discussed symbolic and figural treatments that support students in making use of a pattern to predict further behavior. For instance, some symbolic treatments may facilitate the discovery of a visual structure. A student may investigate The Cross Numbers (Figure 1) by conducting symbolic treatments on the sequence of numbers generated by counting dots in the three first exemplars (5, 9, and 13). Finding that $9 - 5 = 4$ and $13 - 9 = 4$, the student may infer that the sequence increases by 4 for each exemplar. Consecutively, a student may look for ways that 4 is manifested in the figural register: There are 4 arms, and each increase by one dot for each consecutive exemplar. Figural treatments, on the other hand, may provide observable figural structures which students may 'read' directly into the symbolic register. For instance, looking at the figural treatment of $n = 2$ in Figure 2, students may see directly that there are four occurrences of n and a 1 in the center. Without figural treatments like circling or shading dots, 'reading' figural structures into the symbolic register may be more challenging.

2.3 Dynamic learning environments applied in the generalization of figural patterns

Our literature review identified three studies focusing on DLEs applied in the generalization of figural patterns. Healy and Hoyles (1999) analyzed students' learning of mathematical concepts through computer activities. These included a DLE asking students to provide symbolic generalizations of growing matchstick patterns. Analyzing different integrations of computer activities in learning of mathematics, Healy and Hoyles (1999) promoted software that "involves the visual alongside with the symbolic, that is, in which action, visualization, and symbolization are closely interrelated" (p. 83). The study of Pearce et al. (2008) concerned the development of a DLE to support students' generalization. The resulting computer program aimed to facilitate structural algebraic reasoning, where students could explore the concept of variables. Pearce et al. (2008) presented a pond tiling task, asking for the number of $1 \cdot 1$ tiles needed to surround a pond of length l and breadth b. Using the DLE, students could "build with l and b" and visualize different answers, for instance $2l + 2b + 4$ and $2(l + 2) + 2b$. Unfortunately, no empirical studies have been conducted using this computer program.

Investigating pre-service mathematics teachers' technology-aided generalizations, Yao and Elia (2021) included two figural pattern tasks. The study focused on reasoning types, and Yao and Elia showed relationships between naïve empiricism and numerical reasoning on one side and between generic examples and figural reasoning on the other side. However, no further description of the students' work with the figural pattern tasks is included in their article. Yao and Elia concluded that even though the students employed the dynamic properties of digital tools to observe and conjecture, the students struggled to identify the underlying structure and generalize it to broader contexts.

2.4 Knowledge gaps and research question

There is ample evidence for the benefits of figural generalizations of figural patterns and for students' tendency to generalize numerically (e.g., El Mouhayar & Jurdak, 2015; Montenegro et al., 2019). Moreover, Yao (2022) identified transformations that facilitate figural generalizations, but little is known about how to support students in making such transformations. More than 25 years have passed since Healy and Hoyles (1999) concluded that DLEs where "action, visualization, and symbolization are closely interrelated" (p. 83) might provide support for figural pattern generalizations. However, published studies of DLEs supporting generalization of figural patterns either lack empirical data (Pearce et al., 2008) or have not reported on the use of DLE with respect to figural pattern tasks (Yao & Elia, 2021).

In our effort to support students' figural generalizations of figural patterns, we apply the FPapp introduced below. To scrutinize the support provided by this DLE, we use Duval's (2006) framework to analyze the generalization efforts of six pairs of PSTs applying the FPapp. We pose the following research question:

> *What characterizes the pre-service teachers' treatments and conversions as they use the FPapp to generalize figural patterns in task-based interviews?*

3 A dynamic learning environment for figural generalization of figural patterns

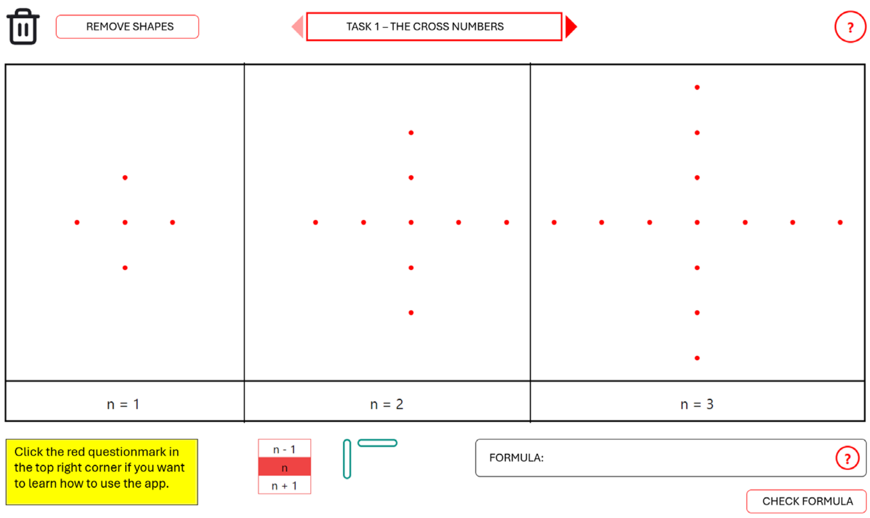
Reviewing teaching practices in algebra, Ellis and Özgür (2024) pointed to the lack of opportunities for students to explore and make connections across representations. The main idea of the FPapp is to support students in making connections between the figural and symbolic registers. This is facilitated through efforts to reduce cognitive complexity related to treatments in the figural register and conversions between the figural and the symbolic registers. Table 1 is a simplified version of the terminology developed by Montenegro et al. (2018), as they applied Duval's (2006) theory to analyze students' work with figural patterns. The first four transformations are ordered chronologically according to a typical solution process when using the FPapp. The fifth transformation, conversions into language, occurs continuously when students collaborate.

Table 1. Transformations conducted by students generalizing and discussing figural pattern tasks using the FPapp. *Transformations facilitated by the FPapp

Transformations	Code	Operationalized to the context of this study
Symbolic-figural conversions*	ScF	Converting from a symbolic expression into a shape applied to the visualized figural pattern
Figural treatments*	Ft	Placing, replacing and removing shapes in the area where figural patterns are visualized
Figural-symbolic conversions	FcS	Converting from a visualized figural pattern (including shapes) into a symbolic algebra expression
Symbolic treatments	St	Symbolic operations (numbers and letters)
Figural/symbolic-language conversions	FScL	Converting from a visualized figural pattern and/or a symbolic expression into verbalized language

The interface of the FPapp is shown in Figure 3. It includes a figural pattern task (exemplified by The Cross Numbers in Figure 3) and tools to facilitate a symbolic generalization.

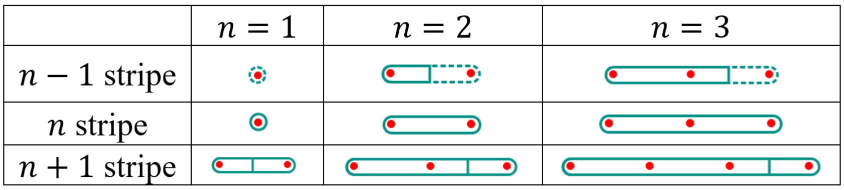
Figure 3. The interface of the FPapp, exemplified by The Cross Numbers



The FPapp utilizes two registers, namely, the symbolic register and the figural register. The former appears in two ways. First, there is an empty formula field where users are supposed to provide a generalized symbolic expression (Figure 3, bottom right corner). This field is initially empty, and the FPapp is designed to determine whether the users' symbolic input is a valid generalized expression of the number of dots in the n^{th} figure. Second, the symbolic register appears in the choice of shape size, where $n-1$, n, and $n+1$ are available options (Figure 3, bottom center). The shapes, on the other hand, belong to the figural register (Figure 3, bottom center). Depending on the task at hand, these shapes may be stripes (vertical or horizontal), squares, or triangles. These shapes are designed to frame dots in the three first exemplars ($n = 1, 2, 3$) of a figural pattern (Figure 3).

As indicated by Table 1, the FPapp facilitates two different transformations. Firstly, by choosing $n - 1$, n, or $n + 1$, where n is the default choice, the app conducts a symbolic-figural conversion. If the user drags a shape (stripe, square, or triangle) into one of the three areas containing a figural pattern specimen, the shape automatically attains the correct size with respect to the value of n. That is; the FPapp dynamically makes symbolic-figural conversions that provide the corresponding shape size. The range of possible horizontal stripes is illustrated in Figure 4. A dashed line framing a dot indicates that the dot is not included by the shape, illustrating the divergence from n. Thus, an $n - 1$ stripe in the $n = 1$ area will appear as a dashed circle only (Figure 4).

Figure 4. The horizontal $n - 1$ stripe, n stripe, and $n + 1$ stripe, as they automatically appear in the $n = 1$, $n = 2$, and $n = 3$ areas



Secondly, the FPapp facilitates figural treatments. In this context, figural treatments mean grouping and regrouping dots in the FPapp interface. The FPapp supports students in covering figural pattern exemplars using different shapes (e.g., stripes, squares, and triangles). This figural treatment is flexible in several ways. All shapes can be removed and replaced at any time, they may overlap, the user may use different shapes to cover the exemplars in $n = 1$, $n = 2$, and $n = 3$, and the user may start over by clicking 'remove shapes' (Figure 3, top left corner). As apparent in Figure 2, different ways of covering figural pattern exemplars make different configurations visually apparent. These may assist students in identifying structures of figural patterns and, subsequently, to generalize the figural patterns using symbolic expressions.

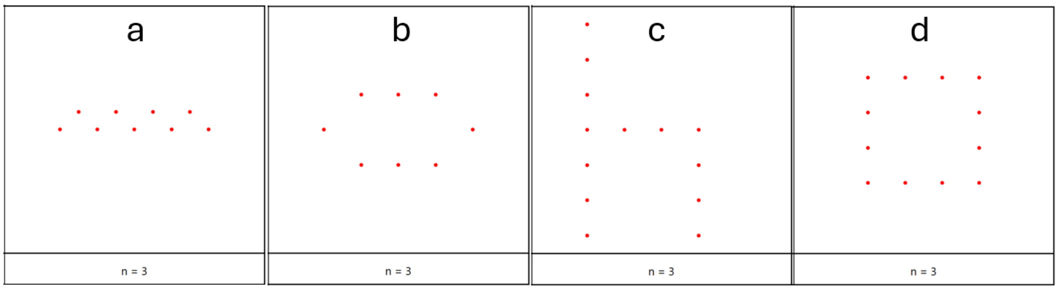
While the FPapp conducts the symbolic-figural conversion automatically, the users must themselves convert the figural structure into the symbolic register. However, as illustrated in Figure 4, the shapes have different appearances, and students may find the symbolic expression of a shape by looking at the shape generator (Figure 3, bottom center). This reduces the cognitive complexity of the figural-symbolic conversion. The symbolic algebra expression is written in the formula field (Figure 3, lower right corner), which is accompanied by a 'check formula' button. By clicking this button, the formula field turns green if the expression is valid and red if it is invalid. The user may then keep treating the figural pattern exemplars or start over by clicking 'remove shapes' in the top left corner (Figure 3).

Two of the transformations in Table 1, namely, symbolic treatments and figural/symbolic-language conversions are relevant in this study but not integrated in the

FPapp. The purpose of not including the opportunity for symbolic treatments is to facilitate figural rather than numerical approaches.

The FPapp contains twelve figural patterns. The Cross Numbers (Figure 1) provides an accessible start, where the arms increasing from one figure to the next are readily identifiable. The Trapezoid Numbers (Figure 5a) and The Oval Numbers (Figure 5b) increase the complexity of identifying the constant and the changing elements. The Chair Numbers (Figure 5c) and The Circumference Numbers (Figure 5d) are included as they facilitate a variety of solutions. The remaining tasks provide experience with squares and triangles as elements in figural pattern tasks.

Figure 5. The $n = 3$ exemplars of The Trapezoid Numbers (a), The Oval Numbers (b), The Chair Numbers (c), and The Circumference Numbers (d)



The FPapp offers a limited range of dynamic elements. The figural pattern tasks are fixed, the possible shapes and sizes are predefined, and the users cannot manipulate the figure exemplars by repositioning dots. This limits the users' opportunities for exploration of the figural patterns. Furthermore, although facilitating figural generalizations, there is no guarantee that students using the FPapp engage in figural reasoning. Thus, there is a risk that they develop practices that reinforce a procedural mindset.

4 Methodology

4.1 Sample, data collection, and analysis

This study was conducted in Norway, where teacher training for primary and lower secondary education is a five-year long master's study. PSTs enroll in programs aimed at teaching in either grade 1 to 7 or grade 5 to 10. All PSTs in the former program and all PSTs specializing in mathematics in the latter program work with figural pattern generalizations at some point in their teacher training. They do so to prepare for the compulsory nationally provided exam in algebraic thinking, which always includes such tasks, even though figural patterns are not mentioned in the 1 to 7 curriculum. The FPapp was developed by Author 2 to offer all PSTs in Norway a free and digitally accessible tool to improve their generalization of figural patterns and, more broadly, their algebraic thinking.

The PSTs in this study are enrolled in the grade 1 to 7 program and attended a 10 credits mathematics course offered in year 3. At the beginning of the semester, the PSTs worked with some figural pattern tasks. The data collection took place three months later. It was initiated by Author 2 informing all PSTs in the course about the ongoing research on the FPapp. Being informed about the implications of attending the study, the opportunity to withdraw, and how data would be stored, analyzed, and presented, volunteers provided written consent before attending the study. Ten female and two male PSTs volunteered.

We wanted to capture the cognitive processes involved as PSTs engage with the FPapp. However, we judged the data generated from having a PST interacting alone with a computer to be insufficient to scrutinize the cognitive processes. We wanted to draw on interplay of signs from different registers, exemplified by PSTs simultaneously gesturing, talking and treating figures using the FPapp. Thus, we designed a context facilitating simultaneous use of language, gestures, typing, and use of computer mouse. The PSTs were assigned work in groups of two to stimulate conversation, and the groups were provided with only one computer to avoid them from working individually. This interplay would support our investigation of the cognitive processes facilitated by the FPapp.

All six interviews were conducted in the same manner. First, Author 2 would direct their attention to one of the figural pattern tasks in the FPapp and ask them to use the shapes to describe the structure of the figural pattern and provide a formula. Author 2 would then remain silent until the PSTs had constructed a symbolic expression validated by the 'check formula' button in the FPapp. Then, Author 2 would lead a conversation with PSTs about their solution using two or more of the questions in Table 2.

Table 2. Questions asked by Author 2 in the intermediate discussions

Q1	Can you write a text to a student who was absent today, explaining the pattern in a way that enables the student to infer how many dots there are in each exemplar?
Q2	Can you explain how the formula relates to the figural pattern?
Q3	Can you use the stripes to explain the structure of the figural pattern?
Q4	Can you explain why the formula will be valid for all exemplars, and not only the ones you see here?
Q5	Can you show that this solution is algebraically equivalent to the previous solutions?

Following the suggestion of Yao (2022), who argued that the inclination to utilize structure is nurtured by identifying multiple generalizations, Author 2 then asked the PSTs to create another generalization of the same figural pattern. At most, he asked for as many as six different ways of generalizing a figural pattern, each time followed by a new discussion. These discussions would sometimes include a request to prove algebraic equivalence with prior solutions using pen and paper. In practice, this implied that the students would use symbolic manipulation to change the new expression into the preceding one.

The six task-based interviews lasted for an average of 70 minutes. About the first 50 minutes were spent on the PSTs going back and forth between solving tasks and discussing their solutions with Author 2. In the remaining part of the interview, the PSTs solved tasks without intermediate discussions with Author 2. In total, the groups created 74 solutions (an average of 12) and engaged in discussions with Author 2 after 48 of these (an average of 8). For instance, Group 1 first made three equivalent symbolic generalizations of The Cross Numbers (Figure 1), before creating four generalizations of The Circumference Numbers (Figure 5d) and three generalizations of The Trapezoid Numbers (Figure 5a). The choice of figural pattern tasks varied from pair to pair, based on Author 2's ongoing interpretation of the PSTs' mastery and occasionally on the PSTs' requests for something similar or something more difficult. The data was collected using a video camera capturing both the PSTs and their gestures aimed at the computer screen. Their digital activity was also captured using a screen recording program.

4.2 Data analysis

The process of data analysis started with the authors viewing the video recordings multiple times to become familiar with the PSTs' communication and interaction with the FPapp. Then, Author 1 transcribed all six video recordings in full, including relevant gestures and digital actions. The PSTs were made anonymous in the transcripts and labelled 'Student 1A', 'Student 1B', 'Student 2A', etc., according to their respective group number. With the transcripts uploaded to a spreadsheet, Author 1 viewed the video recordings and paused for each occurrence of a transformation. The respective code (Table 1) was written in the column next to the corresponding line in the transcript. For each of the six transcripts, a summary was written with respect to the group's performances of each of the five transformations.

The next phase of the analysis was to conduct an inductive analysis examining one transformation at a time. Author 1 would first read the summary of each group with respect to the transformation. Then, all parts of the transcripts where this code had been applied were read again. Author 1 took notes while reading and key episodes were scrutinized by viewing the video recordings repeatedly. Then, to identify the main characteristics of this transformation, Author 1 would reorganize the notes by grouping them thematically. For instance, themes that emerged inductively with respect to the symbolic-figural conversions were 'experimental approach' and 'interpretation issues'. Author 1 then read the reorganized notes again and described the characteristics of the PSTs' performance of this transformation. Finally, both authors revisited the data material and reviewed the final text to ensure that it captured the essence of the data material as validly as possible.

5 Results

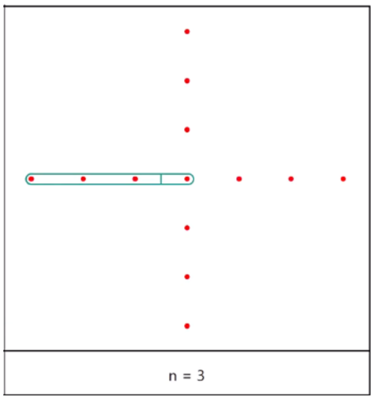
Our research question was *What characterizes the pre-service teachers' treatments and conversions as they use the FPapp to generalize figural patterns in task-based interviews?* We will answer this by presenting the characteristics of the PSTs' transformations, treating one transformation at a time. The presentation will follow the chronologically ordered structure suggested in Table 1.

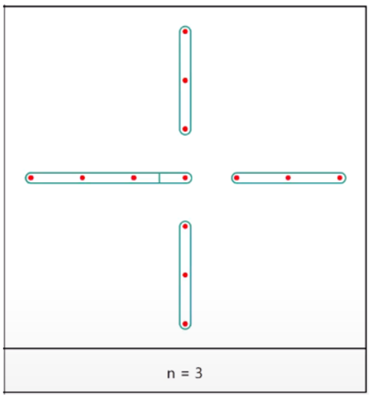
5.1 Conversions from the symbolic to the figural register

In many contexts, students are provided with three exemplars of a figural pattern ($n = 1$, $n = 2$, and $n = 3$). They may approach the task figurally by conducting treatments on the figures, while students approaching the task numerically might start by converting figures into numbers. A unique attribute of the FPapp is the starting point of the students' work. Using the FPapp, the PSTs' starting point was the choice of shapes and their symbolically described sizes (Figure 3, bottom center). The most striking characteristic of the PSTs' symbolic-figural conversions was their inclination to adopt an experimental approach by dragging shapes seemingly at random onto the figural pattern exemplars. One indication of this approach is the fact that 62 of the 74 solution processes were initiated by dragging shapes onto the figures without any prior discussion in the groups about the structure of the figures. Among the remaining 12 solution processes, 9 were initiated by PSTs who had generated new ideas for figural treatments during prior solution processes. In two instances, a full solution of a figural treatment was described before they started dragging shapes. In the final instance, a PST wanted to use an $n + 1$ stripe saying that she wanted to make two shapes overlap, that is, a deconstructive approach (Rivera & Becker, 2008).

With few exceptions, the first attempt at a solution was created using the default n sized shape. The PSTs soon realized how processes starting with this shape most often would result in valid generalizations. In general, placing a random shape often made a structure visually evident, as when the PSTs in Group 2 placed the first $n + 1$ sized stripe on The Cross Numbers (Figure 6). Then, the three remaining n sized 'arms' became readily identifiable.

Figure 6. Placing an $n + 1$ stripe on The Cross Numbers (left) makes the three remaining n sized 'arms' readily identifiable, resulting in the figural treatment shown to the right





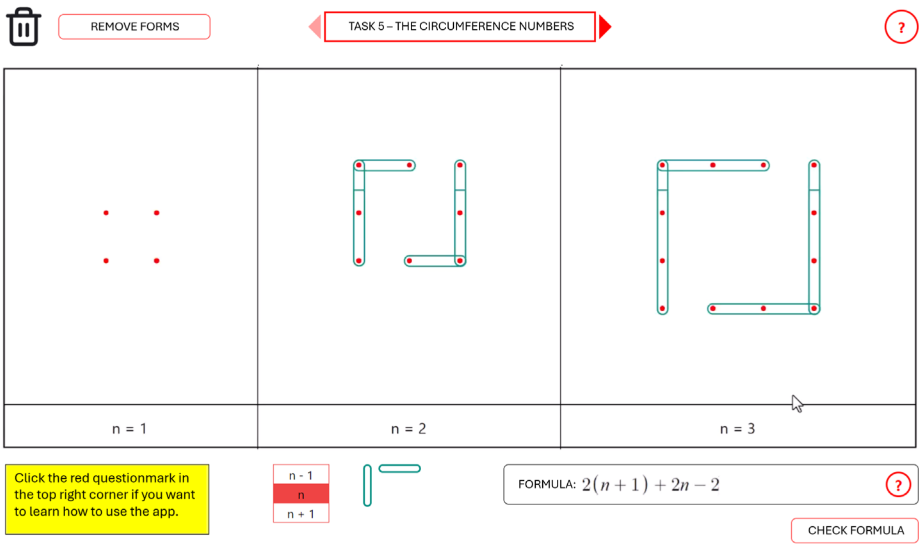
A second indication of the PSTs experimental approach were all the instances where they were asked for a second or third solution. Most often, rather than looking at the figures to identify a new structure, they looked at the shape sizes, pointing out which they had used and which they could try next.

A second characteristic of the PSTs' symbolic-figural conversions was the initial interpretation issues caused by the $n - 1$ and $n + 1$ sized shapes. Some of these were arguably caused by the digital visualization of these stripes (Figure 4). Outside the FPapp, an $n - 1$ stripe on a figure in $n = 3$ would be a stripe of size 2. To support the student in converting this into $n - 1$ after the figural treatments, the stripe was designed with a dashed line including a third dot, to highlight its relation to the n sized stripe. However, some PSTs struggled to interpret the $n - 1$ stripe correctly, resulting in 13 instances of misinterpretations, observed through unproductive figural treatments. Instead of accurately describing the $n - 1$ and $n + 1$ stripes as containing $n - 1$ or $n + 1$ dots, respectively, some PSTs described these stripes as "lacking a dot" or "having an extra dot". However, having to experiment with the $n - 1$ and $n + 1$ sized shapes to create new solutions, the PSTs' interpretation of these shapes evolved. Within the time span of an interview, each group developed a correct interpretation of the $n - 1$ and $n + 1$ sized shapes.

5.2 Figural treatments

The FPapp facilitated effective figural treatments. Shapes were placed, replaced, and disposed of much faster than what is achievable with a pencil and eraser. Initial reluctance to have the shapes overlap or to let dots remain uncovered, disappeared during the interviews. The most remarkable characteristic of the PSTs' figural treatments in this respect, was the fact that they most often conducted treatments on one exemplar only before writing a symbolic expression. In the total of 74 solution processes, all three exemplars were treated in only 18 instances, $n = 2$ and $n = 3$ were treated in 7 instances (exemplified in Figure 7), and $n = 1$ and $n = 2$ were treated in 3 instances.

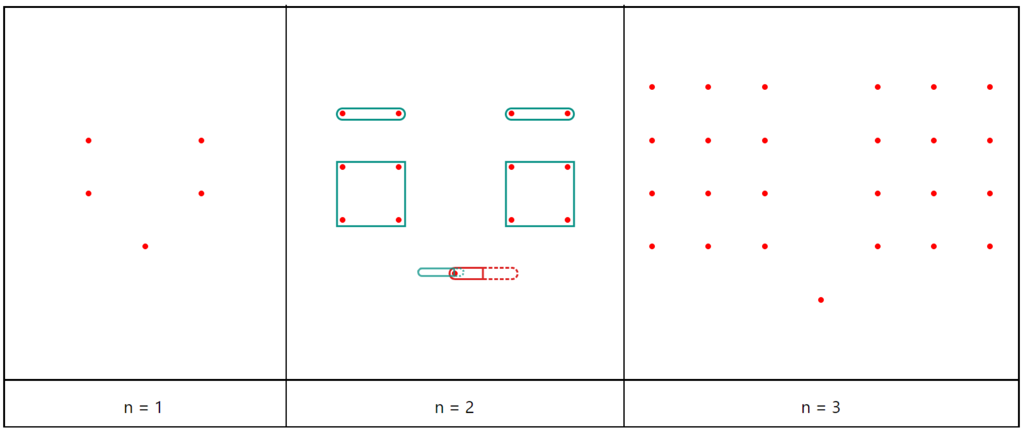
Figure 7. Group 4's figural treatment of The Circumference Numbers. After treating exemplars $n = 2$ and $n = 3$ they wrote the corresponding symbolic expression $2(n+1) + 2n - 2$



Covering only one specimen is a strategy which often provides valid generalizations, but the strategy will occasionally produce invalid generalizations. A generic example describes commonalities shared with all specimens. In the case of linearly growing figural patterns, this means that you need to be able to conduct the corresponding treatments on at least two specimens. Our study reveals two related generalization errors that treating one specimen only might cause.

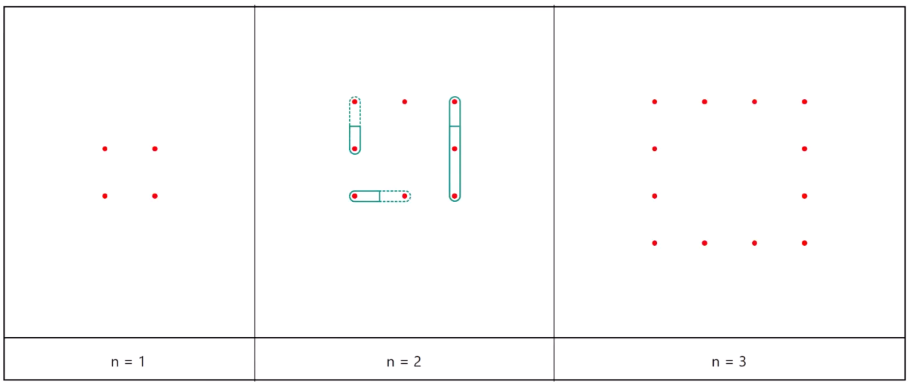
One generalization error was revealed by Group 2 as they were working on The Balance Numbers (Figure 8). Having conducted treatments on $n = 2$, one single dot remained uncovered by a shape. Rather than identifying the dot as a constant term, which would be converted to $+1$, they attempted to cover it with an $n - 1$ stripe. They exemplify a fundamental flaw of treating only one specimen of a figural pattern, namely, risking that constant terms might be treated as variables. An inspection of a second exemplar would reveal this error. We label this error over-letterfication.

Figure 8. Group 2's attempting to place an $n - 1$ stripe on a constant term in The Balance Numbers



We also observed a group almost making the opposite error, namely, under-letterfication. Conducting figural treatments on $n = 2$ only (Figure 9), the PSTs in Group 5 interpreted the uncovered dot as a constant term and suggested a conversion into the symbolic expression $+1$. However, the 'upper edge' is a variable increasing by 1 for each exemplar. This error was discovered by the students as they were typing their symbolic expression into the formula field.

Figure 9. Group 5 treating the $n = 2$ exemplar of The Circumference Numbers, interpreting the uncovered dot in the upper edge as a constant element

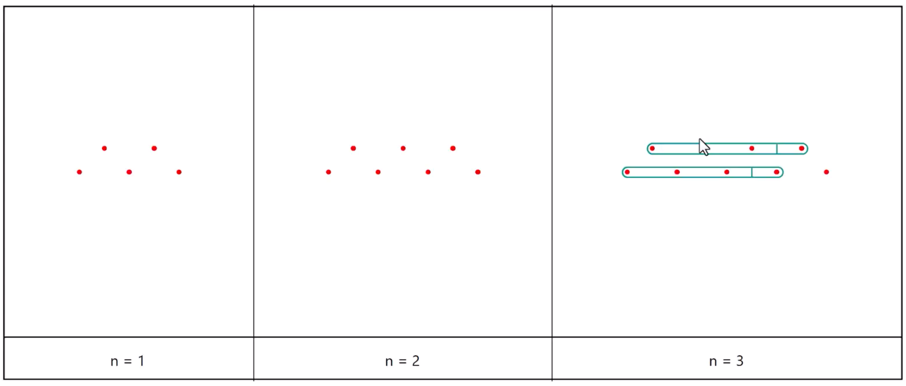


5.3 Conversions from the figural to the symbolic register

The main characteristic of the PSTs' figural-symbolic conversions in this study was their ability to look at the shapes in the figurally treated specimen and 'read' the corresponding symbolic expression. Nowhere in the about seven hours of video recordings did we identify a PST who misinterpreted the symbolic label of a stripe. Arguably, the FPapp reduces the cognitive demand of this conversion due to the different visualizations of the $n - 1$, n, and $n + 1$ shapes, exemplified by the horizontal stripes in Figure 4. Moreover, the symbolic expression of each shape's size is visualized in the FPapp (Figure 3, bottom center).

While the PSTs successfully converted $n - 1$, n, and $n + 1$ shapes into corresponding symbolic expressions, we note that some PSTs occasionally made valid conversions that lacked mathematical efficiency. For instance, Group 1 made three successful solutions where the symbolic expressions imitated their counting procedure. For instance, they covered The Trapezoid Numbers (Figure 10) with two $n + 1$ stripes and left one dot untreated. This was not converted into $2(n + 1) + 1$, but $2(n + 1) - 2 + 3$. We suggest that they interpreted the $n + 1$ stripes as stripes with 'one dot too many', which they had to subtract. Subsequently, they added the subtracted dots in the next term ($+3$).

Figure 10. Group 1's treatment of The Trapezoid Numbers using two $n + 1$ stripes



A second characteristic of these PSTs' figural-symbolic conversions was their immediate identification of invalid symbolic expressions facilitated by the 'check formula' button. Having typed a symbolic expression, the PSTs used this button by default. They were immediately informed if their symbolic expression was invalid, causing them to reevaluate either the figural-symbolic conversion or the preceding figural treatments. Indeed, this feature might potentially contribute to a 'guess-and-check' strategy. Although this strategy might be useful in mathematical problem solving, we do not want students to guess, check, and move on without engaging in algebraic thinking. Students might conduct arbitrary figural treatments on one specimen and succeed in making the figural-symbolic conversions. A verification of their formula may in such instances cause them to proceed to the next task without having engaged properly in the search for commonalities. However, one benefit of this button was evident in our data. Whenever a formula was labelled invalid, the PSTs remained in a productive struggle: They either discussed their figural treatment or their figural-symbolic conversion critically (recorded 14 times) or made a new attempt (recorded 1 time). Without this feature, some might have settled for invalid generalizations and stopped further investigations of the figural pattern.

5.4 Symbolic treatments

The most prominent characteristic of the PSTs' symbolic treatments using the FPapp was the absence of numerical approaches. Occasionally, dots were counted in the discussion an idea, but no solutions were derived from counting dots in consecutive exemplars. Rather, solutions depended on treatments of symbolic expressions converted from figural approaches as they typed a formula in the formula field.

The PSTs were asked to create multiple solutions to some of the figural patterns. In intermediate discussions, Author 2 would ask them to establish algebraic equivalence with prior solutions (Table 2, Q5). Unexpectedly, a second characteristic of the PST's symbolic treatments was their inclination to depend on the establishment of algebraic equivalence to validate new ideas and solutions (recorded 7 times). This treatment was conducted with pen and paper. Five of the six groups made at some point use of one or more symbolic treatments to inspect algebraic equivalence with an already validated answer while solving a task. Most often, it was used to validate suggestions before writing them into the formula field. Group 5 and Group 6 also used symbolic treatments to falsify suggestions. On one occasion, Group 6 took full advantage of the insight that all new solutions needed to be equivalent with the first solution. Working towards a fourth solution to The Circumference Numbers (Figure 5d), Student 6A initiated their work by saying that "you need to end up with $4n$, so you can't have more than four such things", referring to the number of stripes included in a solution. In the subsequent conversation with the teacher, he asked them to show that their fourth solution, $4(n + 1) - 4$, was algebraically equivalent with $4n$. Having conducted the symbolic treatment on pen and paper, the PSTs were still looking down at the paper:

> Yes. It results to $4n$. (Student 6A)
> Because they cancel out [referring to $4 \cdot (+1) - 4$]. (Student 6B)

The teacher then asked the PSTs to describe the structure of The Circumference Numbers in a fifth way. Student 6B kept looking at the formula field and responded within two seconds:

> Yes, we can. Then we'll just do the same [points at $4(n + 1) - 4$ still visible in the formula field]. Four n *minus* one *plus* four. (Student 6B)

The solution suggested by Student 6B was constructed without any explicit reference to the figural pattern or figural commonalities. Knowing that any valid solution needs to be equivalent to $4n$, she inferred this solution purely through symbolic treatment of $4(n + 1) - 4$ into $4(n - 1) + 4$. We see the potential value in performing symbolic treatments on verified solutions to arrive at a new formula. It may help students recognize the mathematical coherence involved in these symbolic treatments ("symbolic treatments do not alter the value") and in these symbolic-figural conversions ("other symbolic expressions convert into other figural generalizations"). However, a symbolic-figural conversion must be conducted for it to be a figural generalization.

5.5 Conversions from the figural and symbolic registers to the language register

The results concerning figural/symbolic-language conversions draw mainly on data from the 48 intermediate discussions, where Author 2 would pose two or three of the questions in Table 2. The PSTs' answers and descriptions rarely concerned either the figural or the symbolic register. Most often, their conversions to the language register included both the figural shapes and symbolic terms.

The first characteristic of the PSTs' figural/symbolic-language conversions was how they tended to respond to questions about the figural patterns' structures. Most often, they merely listed the shapes used in the figural treatment of a specimen of the figural pattern, stating their symbolic expressions. This was the most common response to any of the first three questions in Table 2. Exemplifying this, Group 3 had just created the symbolic expression $3n + (n + 1)$ in generalizing The Cross Numbers (respectively to the configuration shown in Figure 2 for $n = 1$). Author 2 asked them to use the shapes to describe the structure of The Cross Numbers. Student 3B points at the $n = 3$ exemplar:

> In each arm, there are three dots corresponding to the cross number n, which is 3. But in this [points at the $n + 1$ stripe] which is longer, we have added 1. Then you treat that one in a separate term. (Student 3B)

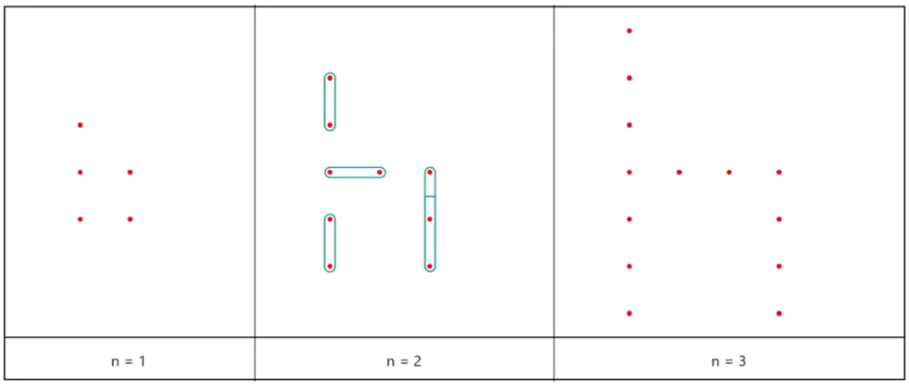
Here, Student 3B answers a question about structure by listing shapes and stating how they convert to the symbolic register. The geometrical shape of the figural pattern and any development between exemplars, which would validate the generalization, remain undescribed. The answer of Student 3B presented here is representative of the PST's answers and descriptions. Rarely would a PST include any reference to the geometrical structure or suggest that a certain collection of shapes would be the same irrespective of which exemplar the PST was treating. Exemplifying a rare exception, Student 1A described their first solution to The Cross Numbers (Figure 1): "It's kind of an addition sign, (…) the figure is shaped like an addition sign. (…) It is an addition sign with four stripes plus one". Conclusively, the most prominent characteristic of the PSTs' figural/symbolic-language conversions, was how figural patterns were described as sets of (randomly ordered) shapes.

Implied above, the second characteristic of the PSTs' conversions into the language register was the lack of attention to how the patterns develop from $n = 1$ to $n = 2$ and from $n = 2$ to $n = 3$. This development is the foundation for generalization statements in this context. Only when asked specifically (Table 2, Q4), they included such descriptions and arguments. Notably, this question was added to the interview guide after the first interview, due to the lack of proper generalization arguments in Group 1. The only generalization statement provided in Group 6 was a student who made the following remark about The Circumference Numbers: "You can think of n as arms, and irrespective of which [figure] it is, it is four arms with the (…) unknown, so it will always be four different arms". Similarly, the students in Group 3 and Group 5 lacked arguments for the generality of their symbolic expressions.

Group 4 was the only group where questions of generality were tackled from the outset. They started by referring to the pattern and argued that the shapes increased by one for each new specimen. For instance, having conducted figural treatments on $n = 2$ exemplar of The Chair Numbers (Figure 11), Student 4A provided the following argument for $3n + (n + 1)$ being a valid generalization:

> We see that in each figure, it increases with the same amount, that is, from 1 to 2 and from 2 to 3 there is a constant increase. So, in number 3, the n will be 3 dots (...) and the same will be true in figure 4, then it will be 4 in n here [points at the 'back' of the chair in the $n = 3$ exemplar], and 4 in n here [points at the left 'leg' of the $n = 3$ chair] and 4 in n here [points at the 'seat' of the $n = 3$ chair], and then you can still use [points at the right 'leg' of the $n = 3$ chair] that $n + 1$ stripe [points at the $n + 1$ stripe in the $n = 2$ chair]. (Student 4A)

Figure 11. Group 4's treatment of The Chair Numbers.



6 Discussion

Now, we will comment on the results of the five transformations involved in the task-based interviews. Again, we follow the order provided by Table 1. Subsequently, we discuss essential insights gained about DLEs aimed at supporting figural generalization of figural patterns.

6.1 Symbolic-figural conversions

Characterizing the PSTs' initial symbolic-figural conversions was the adoption of an experimental approach. Many solution processes were initiated by the seemingly random dragging of an initial shape onto a figural pattern exemplar. Figure 6 exemplifies the effect of an initial placement of a shape, where placing the first shape reveals a structure. Here, we argue that this approach is more likely to happen in a DLE like the FPapp than using pen and paper. Firstly, the FPapp limits the number of possibilities by offering relevant shapes and sizes only. Secondly, the PSTs know that they easily can remove and replace shapes. These two factors combined lower the threshold for taking a first experimental action. It is less likely that PSTs in a pen-and-paper environment would start drawing random shapes on the figure exemplars. Thus, the FPapp restricts its users, making them apply certain shapes. This might support students in experimenting with the figural patterns.

6.2 Figural treatments

Montenegro et al. (2018) argued that conversions between different registers are facilitated by figural treatments that support the identification of commonalities and make them visually available. Our study provides support to the claim that figural treatments facilitate the identification of commonalities. The PSTs applying the FPapp, which was designed to reduce the cognitive complexity of figural treatments, became highly effective in treating figures to visualize different structures. They used both constructive and deconstructive approaches (Rivera & Becker, 2008). However, with respect to their figural treatments, we observed that in 46 of the 74 instances they only treated one figural pattern exemplar before converting to the symbolic register, risking over- and under-letterfication. Indeed, El Mouhayar and Jurdak (2016) considered generic examples as figural generalization. However, even in cases of linearly growing figural patterns, the validity of a generic example rests on identification of commonalities shared with all specimens. We will comment on this later in our discussion.

6.3 Figural-symbolic conversions

The PSTs in our study did not struggle in making conversions from the figural to the symbolic register. We argue that this was partly due to how the FPapp visualizes the commonalities of a figural pattern (as exemplified in Figure 2). Moreover, the FPapp offers a figural-symbolic conversion of each commonality (as exemplified in Figure 4). According to Yao (2022), figural treatments may facilitate the 'reading' of an element into a symbolic register. In our study, considering the PSTs' efficient conversions from the figural register into the symbolic register, it seemed like they were able to 'read' the commonalities directly into symbols.

6.4 Symbolic treatments

Numerical approaches to generalization involve symbolic treatments. Characterizing the PSTs' generalizations efforts was the absence of such approaches. However, symbolic treatments were frequently conducted. The PSTs used algebraic equivalence with known solutions as a guide in searching for new solutions. Thus, like Yao (2022), we identified symbolic treatments that are useful for generalization of figural patterns. In fact, our study provides new examples of how symbolic treatments may facilitate figural pattern generalization. The PSTs in our study conducted symbolic treatments on ongoing solution attempts to establish algebraic equivalence with symbolic expressions of validated solutions. Moreover, we observed an instance where a symbolic generalization of a figural pattern was derived directly from a validated expression through symbolic treatments (Student 6B). Thus, we add to the research field some new 'useful' symbolic treatments. Notably, this presupposes an interpretation of Yao's (2022) term 'useful' as useful to derive valid generalizations, and not necessarily useful to engage in the search for commonalities.

Taking Küchemann's (2010) perspective, where the main reason for solving figural pattern tasks is the attention to the figural register, algebraic thinking plays a limited role in deriving new solutions directly through symbolic treatments. Still, in an initial phase of engagement with figural pattern tasks, symbolic manipulations of validated solutions might be valuable. Students may discover relations between the two registers, as exemplified by Student 6B.

6.5 Figural-language conversions and symbolic-language conversions

The PSTs described figural patterns by listing shapes and their corresponding symbolic expressions. The patterns were most often presented as sets of decontextualized shapes rather than describing the patterns' figural properties and structural development from $n = 1$ to $n = 3$. Most PSTs in this study had a limited repertoire when converting a generalized figural pattern into natural language. While DLEs may support students' generalizations, Yao and Elia (2021) concluded that DLEs may provide students with examples to help them generalize, but that this not necessarily enables them to generalize in terms of general structures. Similarly, the PSTs in our study were supported by a DLE to create generalized symbolic expressions, but they struggled to describe the structure of the figural patterns verbally. Emerging from our results is a critical question: Did the PSTs in our study experience figural pattern tasks as a 'route to algebra', as Radford (2008) suggested they could? Mason (1996) considered generalization acts as the basis for algebraic thinking. Although the PSTs conducted multiple successful generalizations acts, the many cases of imprecise and flawed verbal descriptions of these generalizations cause concern about the algebraic thinking involved. Indeed, the PSTs avoided the numerical approaches criticized by many researchers as limiting the algebraic thinking involved (e.g., Hewitt, 2019; Montenegro et al., 2018). However, our observations indicate that although the PSTs in this study avoided the approach to figural patterns as a set of decontextualized numbers (Küchemann, 2010), some PSTs in essence approached a figural pattern as a set of decontextualized shapes. While the PSTs arguably decomposed figural pattern exemplars (El Mouhayar & Jurdak, 2016), recognized figures as configurations of multiple gestalts (Duval, 1998) and applied both constructive and deconstructive approaches in doing so (Rivera & Becker, 2008), they still struggled to articulate their generalizations. This is untypical for students who apply figural approaches (Yao, 2022). Thus, we hypothesize that some PSTs in this study developed a procedural way of generalizing figural pattern tasks using the FPapp. They developed an efficient procedure to solve the task, but they may have missed the underlying mathematical structure (Yao & Elia, 2021). The presence of this procedure is indicated by the many instances of PSTs treating only one exemplar of the figural pattern, risking errors like over- and under-letterfication. It contrasts with a practice of detecting sameness and difference, which Mason (1996) presented as hallmark for algebraic thinking.

6.6 General discussion, implications, and limitations

The FPapp exemplifies software "in which action, visualization, and symbolization are closely interrelated" (Healy & Hoyles, 1999, p. 83). Students may drag shapes, where the action of dragging a shape into areas representing different values of n cause an immediate change of the size of the shape, and where the appearance of each shape corresponds to a particular symbolic expression ($n - 1$, n, or $n + 1$). As part of a learning activity where looking for structure is decisive (Küchemann, 2010), the effect of the FPapp exemplifies the potential of dynamic (Healy & Hoyles, 1999) and interactive software (Dyrvold & Bergvall, 2023). Like the DLE presented by Pearce et al. (2008), which offered many similar dynamic properties relating action, visualization, and symbolization, the FPapp supports students' algebraic generalizing through a reduction in cognitive complexity. Our study adds empirical support to the assumption that DLEs relating action, visualization, and symbolization may reduce the cognitive complexity of creating figural generalizations.

While the design of the FPapp arguably supported PSTs to derive multiple valid figural pattern generalizations, the results in this study indicate that some PSTs developed strategies that constrained their algebraic thinking: First placing a predesigned shape randomly on a figure exemplar, then placing shapes to circle the remaining dots before typing in the symbolic expressions of these shapes and clicking 'check formula'. If such a formula is verified, the student might proceed without "grasping a commonality" (Radford & Peirce, 2006, p. 5) in the figural pattern. This means that they may conduct figural treatments implying a discursive apprehension of the figural pattern exemplar (Rivera & Becker, 2008) and still not engage in the act of 'seeing' commonalities. Rivera and Becker (2007) described how students would generate numbers from the figural patterns and use these to create guess-and-check abductions. One might draw parallels between this approach and the PSTs in our study placing random shapes on a single figural pattern exemplar and converting the shapes into symbolic expressions. In both instances, students may struggle to explain why the generalization is true.

The results in this study have implications for development of the FPapp. While some properties of the FPapp enable users to develop such a superficial procedure, the same properties may also serve valuable purposes. Without the predesigned shapes, some students may struggle to initiate figural treatments. The dynamically changing shape sizes reduce the cognitive effort of interpreting how a shape looks on a particular figure exemplar, and negative responses from clicking 'check formula' may keep students in productive struggle. Still, the FPapp may be developed to counteract the observed negative effects. First, to emphasize the figural pattern exemplars' commonalities, the FPapp may demand that the user treats all three exemplars with a shape, before allowing the user to add another shape to an exemplar. This way, the user is nudged to discover the figural relationship between exemplars, promoted by Rivera and Becker (2007) as 'figural similarity'. Furthermore, the FPapp may not allow using the 'check formula' button before all three exemplars are treated similarly. Second, while the FPapp facilitates a functional strategy with a figural approach, where shapes are automatically related to the figural step number (El Mouhayar & Jurdak, 2016), a DLE may draw attention to the structural growth by facilitating a recursive strategy. One way of attaining this would be to enable the construction of the $n = 4$ exemplar, where users can construct it by marking dots in a fine-meshed grid. If such a property also encompassed the $n = 1, 2$ and 3 areas, users could apply the strategy of reconfiguring the exemplars. Moreover, the functionality of marking dots in a grid may be extended to enable users to construct their own figural patterns.

The methodological design of this study implies some limitations. As the data collection involves only twelve PSTs, the results and discussions in this study regard a limited population in a confined context. Thus, statements about their performance should not be generalized to the population of Norwegian PSTs, and the representativeness of the characteristics of their transformations is still unknown. Furthermore, we have only analyzed observations and discussions from the PSTs' very first encounter with the FPapp. The long-term effect of the interaction with the FPapp, and the effect of more extensive use, is not considered here.

The PSTs' interactions with the FPapp took place in the context of task-based interviews, where they worked in pairs and a researcher was present. Thus, data collection biases include group dynamics, the collaborative discourse, and the presence of Author 2. The cognitive analysis conducted in this study does not take these factors into account. Future analyses might address how such factors play roles in students' reasoning processes facilitated by the FPapp.

6.7 Conclusions

In this study, we have investigated how we can support students' efforts to create figural generalizations of figural patterns using a DLE. Drawing on video recordings of PSTs solving such tasks using a figural pattern DLE, we conclude that the PSTs effectively constructed multiple valid symbolic generalizations. We ascribe this success to the ways that the DLE reduced the cognitive complexity of the transformations involved (Duval, 2006). The PSTs adopted experimental approaches, developed strategies for figural treatments, converted these to the symbolic register, and conducted symbolic treatments along the way to search for equivalence with validated solutions. Numerical approaches were absent.

Secondly, we demonstrate that students may apply a DLE to create valid symbolic generalizations and still reveal shortcomings in their developing algebraic thinking. Some of the PSTs in our study employed the FPapp to develop a procedure where they created valid symbolic expressions without necessarily searching for commonalities or engaging in algebraic thinking. We conclude that although DLEs may facilitate students' figural approaches to generalization of figural patterns, students may utilize DLEs in ways that hinder algebraic thinking. DLEs, exemplified by the FPapp in this study, benefit from being scrutinized to uncover shortcomings and unintended use. This may result in improvement of the DLEs and an increased attention to practices students develop through their interaction with DLEs.

We encourage future research on the treatments and conversions involved in generalization of figural patterns. Valuable contributions to the research on DLEs and figural pattern generalization include scrutinizing attempts to treat the shortcomings discussed in this article. Moreover, we encourage research on the effects of more extensive use of such DLEs.

Research ethics

Author contributions

J.S.: Methodology, data transcription, conceptualization, analysis, theoretical and analytical discussions, writing–original draft preparation, writing–review and editing.

C.A.L: Funding acquisition, development of FPapp, methodology, data collection, theoretical and analytical discussions, manuscript review and editing.

Both authors have read and agreed to the published version of the manuscript.

Artificial intelligence

The use of Artificial Intelligence (ChatGPT v.3) in this research is limited to requests for synonyms, translations of Norwegian phrases into English, and minor linguistic refinements.

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Author 2 provided financial support for the technical development of the Figural pattern Computer Application. However, no funding was received for conducting this study.

Institutional review board statement

This research project, including data collection and data handling, was approved by Norwegian Agency for Shared Services in Education and Research.

Informed consent statement

Description of the participants' consents is provided in the Methods section.

Data availability statement

Data is unavailable due to privacy restrictions in line with the guidelines from Norwegian Agency for Shared Services in Education and Research.

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The authors thank the PSTs who participated in the data collection.

Conflicts of interest

The Figural pattern Computer Application applied in the study is a non-profit tool available free of charge and without advertisements. The authors have no financial or non-financial interests to disclose.

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