Supporting argumentation in mathematics classrooms: The role of teachers' mathematical knowledge

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Reform movements in mathematics education advocate that mathematical argumentation play a central role in all classrooms. However, research shows that mathematics teachers at all grade level find it challenging to support argumentation in mathematics classrooms. This study examines the role of teachers' mathematical knowledge in teachers' support of argumentation in mathematics classroom. The study addresses a documented need for a better understanding of the relationship between mathematical knowledge for teaching and instruction by focusing on how the knowledge influences teachers' support of argumentation. The results provide insights into particular aspects of teachers' mathematical knowledge that influence teachers' support of students' development of valid mathematical arguments in mathematics classrooms and suggest implications for research and practice.

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1 Introduction

Reform movements in mathematics education advocate that mathematical argumentation play a central role in all classrooms. In particular, mathematics classrooms should become communities of inquiry in which students seek, formulate, and critique the validity of each other's conjectures and arguments (See e.g., National Council of Teachers of Mathematics [NCTM], 2000; CCSSM, National Governors Association Centre for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). Yet, research shows that teachers find it challenging to support argumentation in mathematics classrooms (See. e.g. Ayalon & Even, 2016; Bieda, 2010). Furthermore, teachers' mathematical knowledge plays an important role in their support of this practice in mathematics classrooms (Cengiz et al., 2011; Yackel, 2002). The study examines aspects of teachers' mathematical knowledge that influence teachers' support of argumentation. There is a documented need for a better understanding of the relationship between teachers' mathematical knowledge and aspects of instruction (See e.g. Cengiz et al, 2011). This study examines the relationship between mathematical knowledge and teachers' support of argumentation in classrooms. Processes involved in argumentation are similar to those involved in mathematical thinking. Therefore, supporting argumentation is supporting mathematical thinking, the topic of this special issue.

2 Theoretical Background

2.1 Argumentation in mathematics teaching

In the mathematics education community, argumentation is considered an important disciplinary practice that should be promoted in all classrooms. The *Principles and Standards for School Mathematics* of the National Council of Teachers of Mathematics (NCTM, 2000) emphasize reasoning, proof, and communication, three essential components of argumentation. The Common Core State Standards for Mathematics (CCSSM, 2010) state that students should be able to "Construct viable arguments and critique the reasoning of others" (p. 7). There are several reasons for promoting argumentation in mathematics classrooms. Students' ability to justify claims, which is part of argumentation, is considered a key indicator of students' mathematical thinking (CCSSM, 2010). Argumentation is a natural part of doing mathematics since mathematics is a proving science and mathematical argumentation is central to proving (Ubuz, Dincer, & Bulbul, 2012). Argumentation can also help promote equitable learning opportunities in classrooms. This is because argumentation is a central construct to discourse and classroom discourse influences students' access to mathematics. Teachers can promote equity in learning by providing all students with opportunities to produce and defend their arguments in classroom discussions (Bieda, 2010).

Research on argumentation in mathematics classrooms has examined the classrooms conditions and the role of the teacher in facilitating the process (Ayalon & Even, 2016; Conner et al., 2014; Douek, 1999; Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Maher, 1998; Mueller et al., 2014; Yackel, 2002). This research shows that teachers can play a central role in supporting argumentation. They can negotiate classroom norms that foster argumentation as the core of students' mathematical activity, support students as they interact with each other to develop arguments, and supply argumentative supports (data, warrants, and backing) that are either omitted or left implicit (Yackel, 2002). When supporting students working collaboratively to develop mathematical arguments, teachers can prompt students to establish claims and justifications, encourage them to critically consider different arguments, present to students what constitutes acceptable mathematical arguments, and model ways of constructing and confronting arguments (Ayalon & Hershkowitz, 2017).

Despite its importance for mathematical learning, the implementation of argumentation in mathematics classrooms is not common practice (Bieda, 2010; Bleiler, Thompson, & Krajcevski, 2014; Staples, Bartlo, & Thanheiser, 2012). Research shows that teachers find it challenging to incorporate this practice in classrooms (Ayalon & Even, 2016; Bieda, 2010). They find it challenging to engage students in constructing and critiquing arguments (e.g., Ayalon & Even, 2016) and their interpretations of facilitating argumentation may not be aligned with those of reformers such as assuming that mathematical argumentation can occur with relatively little scaffolding by the teacher (Kosko et al., 2014). There is a general consensus that research on teacher support of argumentation is still in its infancy and more needs to be known about teacher knowledge and practice of argumentation (Kosko et al., 2014; Mueller et al., 2014). This study addresses this issue by examining aspects of mathematical knowledge that influence teacher's support of argumentation.

2.2 Mathematical argumentation

Research on argumentation in educational settings frequently uses Toulmin's (1969/2003) scheme of argumentation as an analytical tool. According to this scheme, the core of an argument consists of three essential parts: *claim*, *data*, and *warrant*. The *claim* is the assertion of which an individual is trying to convince others. The *data* are the evidence that the individual presents to support the claim. The *warrant* is the explanation why the claim follows from the data. Members of a group may not be convinced that a claim follows from the data and question the validity of the warrant. In such cases, the individual may present a support or *backing* for the warrant. The model has two additional components: a modal *qualifier,* which refers to the degree of confidence about a claim, and a *rebuttal*, which refers to the conditions under which the conclusions may or may not hold. The restricted version of Toulmin's scheme is considered sufficient to analyze arguments at school level (Knipping and Reid, 2015; Krummheuer, 1995). However, Inglis et al. (2007) showed that considering the additional components can provide a more comprehensive description of individuals' argumentation and reasoning processes and helps investigate arguments similar to those made by mathematicians.

Krummheuer (1995) extended Toulmin's notion of argumentation from an individual to a collective notion by distinguishing between situations where one individual tries to convince an audience about the validity of a claim and situations where two or more individuals interact to attempt to establish a claim, which Krummheuer called collective argumentation. Collective argumentation thus becomes an interactional discursive accomplishment and an argument can no longer be analyzed solely by considering a sequence of statements that are made. The functions that various statements serve in the interaction of participating individuals become critical to making sense of the argumentation that develops. What constitutes data, warrants, and backing is no longer predetermined, but rather negotiated by the participants in the interaction. This makes collective argumentation a useful construct for analyzing mathematical activity characterized by collective problem solving (see, e.g., Whitenack and Knipping, 2002; Van Ness and Maher, 2019). In particular, this makes collective argumentation a useful construct for analyzing the teacher's role in facilitating argumentation as the teacher interacts with students to support the development of valid mathematical arguments (Yackel, 2002). In this study, teachers' support of argumentation refers to teachers' discursive role in supporting students' development of valid mathematical arguments to support their solutions as they work collaboratively on challenging mathematical problems.

2.3 Mathematical knowledge for argumentation

It is generally accepted in the mathematics education community that the quality of mathematical teaching depends on subject-related pedagogical knowledge that teachers bring to bear on their work and this type of knowledge goes beyond what one acquires as a student of mathematics (Adler & Davis, 2006; Ball et al, 2004; Ball, Lubienski, & Mewborn, 2001). However, there is no universal agreement on one widely-accepted framework for describing this knowledge (Petrou and Goulding, 2011). Several conceptualisations or models have been proposed over the years (See e.g., Shulman, 1986; Fennema and Franke, 1992; Rowland, 2005; Rowland, 2007; Rowland, Huckstep, & Thwaites, 2003). Petrou and Goulding (2011) provide a comprehensive review of the models focusing on their meaning, importance, limitations, implications for research and teacher development, and the political context in which they were developed. They note that the models elaborate rather than replace Shulman's (1986) well-known conceptualisation of content-related categories of teacher knowledge, particularly the categories of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK).

One model is the Mathematical Knowledge for Teaching (MKT) framework proposed by Ball et al. (2008). The model distinguishes among three_SMK subcategories. *Common Content knowledge (CCK)* is the mathematical knowledge held by people who have not taught children mathematics. *Specialized Content Knowledge (SCK)* is the mathematical knowledge specific to teaching and includes being able to examine alternative representations, provide explanations, and evaluate unconventional methods. *Knowledge at the Mathematical Horizon* is the "awareness of how mathematical topics are related over the span of mathematics included in the curriculum." The MKT framework also distinguishes among three PCK subcategories. *Knowledge of Content and Students (KCS)* is knowledge of how students learn specific mathematical ideas and concepts, students' common conceptions and misconceptions, and what students are likely to do in specific mathematics tasks. *Knowledge of Content and Teaching (KCT)* is knowledge of effective strategies for teaching particular content, and includes useful examples for highlighting important mathematical issues, and the advantages and disadvantages of using particular representations to teach specific ideas. There is also *Knowledge of Curriculum (KC)* which is provisionally placed in PCK category.

In this study, teachers' mathematical knowledge refers to MKT knowledge for supporting argumentation. Research shows that having strong knowledge in MKT areas enhances teachers' support of students' mathematical learning (Hill et al. 2005, 2004). However, the relationship between knowledge in MKT areas and instruction remains unclear (Ball et al, 2001; Cengiz et al. 2011; Tirosh and Even 2007). This study examines the relationship between teachers' MKT knowledge and teachers' support of argumentation in mathematics classrooms. The focus is on identifying aspects of MKT in the areas of SCK, KCS, and KCT that help teachers support students' development of valid mathematical arguments. The following research questions guided the study:

1. What aspects of mathematical knowledge for teaching (MKT) in the areas of SCK, KCS, and KCT support teachers in facilitating students' development of valid mathematical arguments in collaborative problem solving?
2. How do such aspects support teachers in promoting argumentation in mathematics classrooms?

3 Method

3.1 Research context

The three-year after-school classroom-based *Informal Mathematical Learning* project (IML) provided the context for the present study. The goal of the project was to understand how students reason in building mathematical knowledge as they worked collaboratively on challenging mathematical tasks. The project was implemented in an economically depressed urban district in the Northeast coast of the United States. Ninety-eight percent of the students were African American or Latin. Approximately twenty-four sixth-grade students, all African American or Latin, volunteered to participate in the project. During IML research sessions students worked for sixty to ninety minutes on mathematical tasks selected from several mathematical content strands including combinatorics, proportional reasoning, early algebra, and probability with dynamic software. Students worked in particular conditions: they were encouraged to work collaboratively and to always justify their solutions to problems to each other. Their contributions were encouraged and always received positively. They were asked to evaluate their claims based on whether or not they were convinced that they “made sense” and they were given extended time to work on tasks. Follow-up interviews with students were conducted after sessions to gain an in-depth understanding of the students’ reasoning.

Seven elementary school mathematics teachers participated as interns in the IML project. Their participation was part of a professional development program designed to help teachers develop knowledge to promote mathematical reasoning and justification in teaching. During the first year of the project, the teachers observed researchers lead research sessions with a class of sixth-grade students. During the second year, partner teachers led similar sessions with a new cohort of sixth-grade students, implementing the same content, while other teachers, researchers, and graduate students observed. At the end of each paired teacher implementation session, one-hour debriefing meetings were held for reflection and discussion of challenges in supporting students’ thinking.

3.2 Data source

All research sessions and debriefing meetings in the IML project were videotaped and digitized. Several cameras captured students’ mathematical activity in small groups and whole class discussions as well as teachers or researchers' exchanges and interactions with students and facilitation of conversations about students' presentations on an overhead projector for sharing of student work. One camera captured the debriefing meetings. Data for this study was selected from videos of IML student sessions led by teachers in the second year of the IML project and of debriefing meetings held at the end of the sessions and attended by teachers and researchers. Examining videos of IML sessions showing teachers' pedagogical actions and videos of debriefing meetings showing teachers reflecting on their actions is consistent Shulman's observation that the knowledge base for teaching is distinguished by "the capacity of a teacher to *transform the content knowledge he or she possesses into forms that are pedagogically powerful*" (1987, p. 15; emphasis added) and the distinction by Ball (1988) between knowing mathematics '*for yourself* 'and knowing in order to *be able to help* someone else learn it (emphasis added). This suggests that mathematical knowledge for teaching is reflected both in teachers' utterances/reflections (Debriefing meetings) as well as their actions (IML sessions) while teaching. There were approximately twenty IML sessions and an equal number of follow-up debriefing meetings during each year of the project. Data for this study consisted specifically of videos of six teacher-led sessions and debriefing meetings held at the end of the sessions, all involving versions of the Tower Problem, a task that was part of the counting strand. The statement of the Four-Tall Tower Problem when choosing from two colors read as follows:

> You have two colors of Unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates.

Other versions of the Tower Problem used in the IML project included the Two-tall tower problem when choosing form three colors and the three-tall tower problem when choosing from three colors. The tower problem is reasoning-rich. Students often use different strategies and types of reasoning to solve the problem (See e.g., Maher et al, 2010). This and the fact that in IML students were asked to justify their solutions to each other and to teachers/researcher helped create a learning environment for studying teacher support of argumentation. This is a case study. Stake (1994) defines an instrumental case study as a form of research where "a particular case is examined to provide insight into an issue or refinement of theory." The six teacher-led student sessions and corresponding debriefing meetings involving versions of the Tower

Problem were (the instrumental case that) was examined to gain insight into aspects of mathematical knowledge for teaching that help teachers support argumentation in mathematics classroom (issue of interest).

3.3 Analysis

Data analysis combined video analysis methodologies (see, e.g., Powell, Francisco and Maher, 2003; Erickson, 2006) and analytical approaches for studying argumentation (See e.g., Krummheuer, 1995; Knipping et al, 2015). The analysis had two parts corresponding to the two types of data used in this study: (1) analysis of the IML teacher-led sessions and (2) analysis of the debriefing meetings that followed the sessions. In both cases, the analysis involved several iterations of three sequential and interrelated main steps. First, all videos were viewed several times to have a sense of the data as a whole. Second, the videos were viewed again and parsed into episodes. Third, all episodes were analyzed for insights into aspects of teachers' MKT knowledge that support teachers' actions to promote argumentation. In the case of IML sessions, the episodes consisted of instances of sustained interaction between teachers and students where the teachers tried to support students in establishing claims. In the case of debriefing meetings, the episodes were instances in which teachers reflected on their interventions during IML sessions. Analysis of the episodes from IML sessions involved (1) coding students' developing arguments using Toulmin's model, (2) open coding for aspects of mathematical knowledge for teaching in the areas of SCK, KCS, and KCT reflected in teachers' actions to support argumentation and (3) describing how those aspects influenced teachers' support of argumentation, particularly in responding to or eliciting valid mathematical arguments supported by those aspects. The challenges of using the MKT framework for characterizing teachers' knowledge base have been documented in the literature. Cengiz et al (2011) found it difficult to distinguish between CCK and SCK and chose to collapse the two categories into one category: Common Content knowledge (CCK). Similarly, Petrou and Goulding (2011) noted it may be difficult to distinguish between SCK and PCK in the MKT framework. Also, the Mathematical Horizon and Knowledge of Curriculum (KC) domains remain under-conceptualized and require further refinement and investigation (Ball et al., 2008; Lesseig, 2016; Petrou and Goulding, 2011). For this reason, the study focused on the three categories of SCK, KCS, and KCT and defined them as follows:

1. SCK – knowledge of argumentation as a mathematical process, including its components, structure, and function (e.g., Toulmin's scheme, types of arguments, valid and invalid argument and functions and roles of arguments)
2. KCS - knowledge of students' typical conceptions and misconceptions as well as what they can do when engaging in argumentation (e.g., typical Harel and Sowder's (2007) proof schemes that student may use to determine if an argument is convincing or not)
3. KCT – Knowledge of interventions for (1) eliciting and (2) responding to students' arguments (e.g., how to help students transition from authoritarian or empirical justification toward more analytical types of arguments; how to challenge invalid arguments; how to support generalization of arguments)

Analysis of debriefing meetings was used to corroborate the analysis of IML sessions. The analysis of both kinds of data helped get a more accurate interpretation of teachers' actions for supporting argumentation. All coding and interpretations were discussed within a research team until disagreements were resolved to enhance reliability.

4 Results

Data analysis revealed several aspects of teachers' mathematical knowledge for teaching that support argumentation. These are described below along with how they influenced teachers' support of students in building valid mathematical arguments.

4.1 Argumentation as a discursive activity

In the episode below students were working on finding towers two-tall with exactly two colors, blue (B) and yellow (Y). Martina was working with two other students. She built four towers [BY, BY, YB, YB] and continued to build more towers despite having duplicates. When the teacher asked her how many towers there were in total, Martina said, "It depends on how many blocks [sic unifix cubes] you have." This prompted teacher to intervene:

Marina	Did you say two colors?
T1:	Two different colors, two tall. How many towers can you build?
Martina:	I guess it depends on how many blocks [sic unifix cubes] you have
T1:	Well, okay. Suppose you had more blocks? Here is another one (points at a tower the student had built). You built that one. (Asks all students) What happens if she builds that one?

Students 1 and 2: They are all the same.
Student 2: (Talking to Martina) You gotta take one of each of them out, like this. (removes duplicates from Martina's towers and leaves only the towers BY and YB)
T1: So, you can only make two different towers, two colors, two tall. [to all students] Do you agree?
Student 1 and 2:Yes.
T1: (Asking Martina) Do you agree, Martina? (Martina nods). Right. Because what happens? Even if you had more blocks, what happens?
Martina: It is still going to be the same.
T1: Correct. So, you start building the same thing. So, it's a repeat. Good.

Using Toulmin's scheme, Martina's argument can be coded as follows: any two cubes stuck together make a tower and there can be least four towers (data). Since more towers can be built if more cubes are available (Implicit Warrant), the total number of towers that can be built depends the number of unific cubes available to choose from (claim). Martina's argument is not valid because the data in her argument includes duplicate towers, which is not consistent with the specifications of the problem since it requires that she builds *different-looking* towers. The teacher successfully challenges Martina's argument and two moves were crucial in her intervention and provide nights into the role of MKT in supporting argumentation. First, the teacher tells Martina to pretend that she has as many towers as she wants and, as Martina tries to build more towers, the teacher points at duplicates in Martina's set of towers ("Here is another one", "You built that one"). Second, the teacher tries to involve the other students in the group in examining Martina's argument by asking questions not only Martina but also to the students (e.g., "What happens if she builds that one?" and "Do you [Martina] agree [with *them*]?"). The two moves provide insights into the role of MKT in supporting argumentation. The first move shows that the teachers understood that Martina's argument is not valid because it includes incorrect data (SCK). However. this was not enough to help Martina realize the mistake in her argument. It is the second move that effectively helps Martina realize the mistake in her argument as the other students in the group tell Martina that her set of towers includes duplicates ("they are all the same") and one student even removes the duplicates from her set of towers. This highlights the importance of the view of argumentation as a social practice which emphasizes social and cultural aspects and persuasion as its main function, compared to a view of argumentation that emphasizes structural or cognitive aspects and validation as the main function of the process (SCK). In this episode, the view of argumentation as a social practice allowed the teacher to use students' collaboration in the evaluation of a mathematical argument as a strategy for supporting argumentation (KCT). The strategy helped Martina realize abandon an invalid argument.

4.2 Counterexamples to challenge arguments

In IML sessions, students eventually arrived that at the correct solution that there are in total sixteen towers four-tall when choosing from two colors. The teacher challenged the students to justify the solution. One student, Gabriel (Gabe) built four groups of four towers and then said that there are sixteen towers in total because "four times four is sixteen," (See Figure 1). When the teacher asked the student why he said "four times four?" the student said "you can divide the sixteen towers into groups of four towers each." The teacher was not convinced by the student's explanation, but did not know how to challenge it and walked away. In the debriefing meeting that followed the session, the teacher shared with the audience her difficulty in challenging the 4x4 argument admitting that she did not know how to "elicit the convincing argument" from the student:

> T1: Gabe said, "Sixteen divided by four is four." I am like, "Well, what does that have to do with what we are doing?" *So, the question I have as a facilitator is, what do I do in order to elicit the convincing argument?* Because even with Yonnie, he is getting at a point where he is getting annoyed with me because I keep saying, "How do you know?"

One teacher in the audience suggested asking the students to write their solutions on posters and then share them and discuss in class. However, the researcher who was facilitating the meeting proposed a different idea. She suggested first asking the students to predict how many towers there would be three-tall when choosing from two colors and then asking them to investigate empirically if their prediction was correct. The researcher explained how she thought the students' reasoning would unfold. The students would predict nine towers by analogy with the 4x4 argument and then, when trying to build them, they would not find nine towers and would find only eight. They would also notice that every tower has an opposite-looking tower and would conclude that the total number of towers had to be even and would abandon their prediction:

> Researcher: The way I frequently address that is to say, "*Okay, how many [towers] do you think there would be if they were just three tall with two colors to choose from*?" And Yonny [student] is going to say "Nine." And I say, "Hm ... that does work with your prediction. How are you going to test that one out?" And Yonny is going to say, "I guess I can build them." Then I say, "But don't

> mess up your fours [four-tall towers they built].” Make sure they don’t destroy their four ones in order to do the three ones. And then when they can’t find them [the nine predicted towers] ...there is a little bit of disequilibrium...Also, “what do you think it’s going to be for five?” Then they’re going to say “Twenty-five.” *Many people say, you know, “We know it’s going to be even. So, it can’t be nine. So, it must be one less.” People of all ages stay with the four by four but modify it because we know there are opposites. It’s got to be an even number of them. Eight is one fewer than nine.* So, it’s got to be twenty-four. I would keep pushing them in all the ways you are thinking about. That is just my suggestion.

In the following session, the teachers implemented the researcher’s suggestion and events unfolded exactly as the researcher had predicted. Several students predicted that there would be “nine” towers three-tall when choosing from two colors. One student, Mohamed, said, “Maybe nine because three times three equal nine.” However, the students could not find 9 towers. They found only eight towers. Also, Martina, the student in the previous episode, noticed that every tower had a “double” and concluded that that there could not be nine towers because the total number of towers had to be an even number:

> Martina: I said if it was 9 there would be like double of them because of the opposite of one another. Like this one blue (BBB) and this one is yellow (YYY) there would be another one. Except there would be an opposite. So, it has to be an even number.

Based on Toulmin’s scheme, Gabe’s argument can be coded as follow: For towers <u>four-</u>tall there are <u>four groups</u> of <u>four towers</u> each (data). Therefore, the total number of towers must be sixteen towers (claim) because the total number of towers must “height x height” (implicit warrant). The implicit warrant in Gabe’s argument is supported by the students’ prediction that there would be nine towers three-tall when choosing from two colors because “three times three equal nine,” where “three” is the height of the towers. However, is not valid as a general warrant as it did not work for the three-tall towers problem when choosing from two-colors. This episode highlights the importance of counterexamples in supporting argumentation. The three-tall towers problem when choosing from two colors served as a counter-example to the “height times height” general warrant implicit in the “4x4” and “3x3” arguments. If the warrant was correct, there would be 3x3=9 towers three-tall when choosing from two colors. However, the students could not find nine towers and could not support the “height x height” warrant implicit in the 4x4 argument. Knowledge of counterexamples that challenge particular arguments can be considered an example of SCK. This episode shows that such knowledge can help support argumentation by challenging invalid mathematical warrants (KCT).

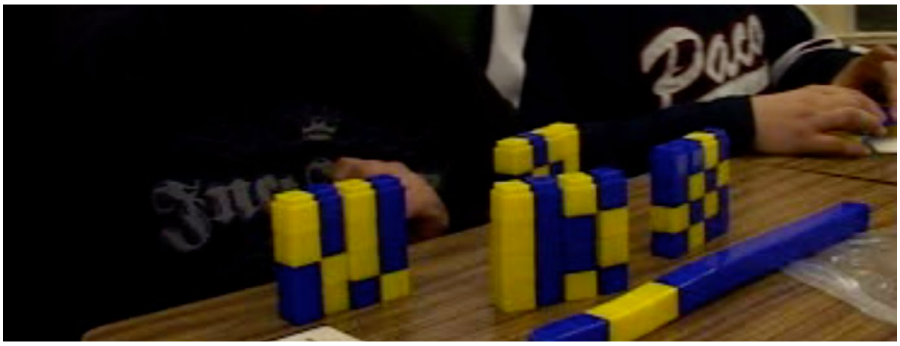

Figure 1. Gabe (on left) built for groups of four towers each when choosing from two colors

4.3 Knowledge of Students' argumentative strategies

In the 4x4 argument above the researcher introduced her suggestion by saying "*The way I frequently address that is to say…", "Yonny is going to say…,"* and "*Many people say…"* This suggests that the researcher was using her knowledge of how students reason when working on the tower problem to come up with the suggestion. The researcher knew that students often came up with invalid warrants such as the "height x height" embedded in the 4x4 argument and designed interventions to challenge them using counterexamples such as the three-tall tower problem when choosing from two colors. In contrast, in another episode in which students were asked to build towers three-tall when choosing form three colors, Yonny, the student mentioned in the 4x4 argument above, used a reasoning-by-cases strategy and a "diagonal strategy" to prove that he has built all towers within the cases (See diagonals in Figure 3). An example of the application of the diagonal strategy to prove that all towers with three reds and one yellow (3R and 1Y) have been found is to show that the yellow cube has occupied all possible positions in the tower forming a (yellow) diagonal. Teacher T5 explained during the debriefing meeting that he was familiar with the use of the strategy when building towers from two colors, but was surprized to see it being used with towers with three colors:

> T5: I don't know why in my mind I didn't think it would work when I went around to see his. At first, I didn't say anything to him. I've learned that. But I just looked at it and asked him to explain it, but now it makes sense.

The statement suggests that not knowing that the diagonal strategy could be used with towers with three colors constrained the teacher's support of Yonny's reasoning process (*I didn't say anything to him…. But I just looked at it and asked him to explain it)*. This and the way the researcher introduced het suggestion in 4x4 episode highlight the importance of KCS in supporting argumentation. It shows that having knowledge of argumentative strategies that students are likely to use when making argumentation (KCS) can help teachers design effective strategies for support argumentation. In the 4x4 argument, knowing that students could use the "height x height" argument helped the researcher come up with a counterexample to challenge the argument. However, T5 did not know that students could use the diagonal strategy with three-color towers and this limited the teacher's ability to support a student in developing of a valid argument based on this strategy. Overall, knowledge of students' typical reasoning or argumentation support strategies for promoting argumentation that build on students' argumentative reasoning (KCT). The three-tall tower problem when choosing from two colors was carefully designed task to be a counterexample to the "height x height" argument in 4x4 episode.

4.4 Representation

While working on finding all towers 3-three cubes tall when choosing from three colors, Yonny came up with the tower arrangement displayed in Figure 2. The arrangement shows "opposite" groups of towers (i.e., towers in one group are opposites of towers in the other group) and diagonal lines in different colors running through all groups except the groups of single-color and three-color towers. Yonny told the teacher that there were 27 towers in total because "I can't find [any more of] them." The teacher said, "that's not a proof" and Yonny responded, "I used opposites" and explained his idea using the diagonal lines:

T1:	Wait. What do you mean? So, these are opposites?
Yonny:	Yeah.
T1:	*Explain it to me why?*
Yonny:	Because like I said before. You got the *yellow in a little line* here [traces the yellow diagonal in towers 2R1Y]. You got the *red in the little line* here [traces a red diagonal in the opposite group of towers]
T1:	*What do you call that line?*
Yonny:	*A diagonal line*
T1:	Ok, so you are saying there is a yellow in this diagonal [towers 2R1Y] and a red in this diagonal [towers 2Y1R]? So, what does that mean?
Yonny:	*They are opposites*. So, you got yellow and the red on both sides [of the diagonal]

The teacher turned her attention to the pairs of towers with three colors in the arrangement. The teacher started to put the towers together and then stopped and asked if the towers could form a group. Yonny said "No. because there is no other similar [*sic* opposite group]," indicating that the group would not have an opposite group. The teacher then turned her attention to the diagonals and pointed out that there were no diagonals in the group of towers with three colors. Yonny stared at the towers for a little bit and then all of a sudden had an "aha" moment. He reorganized the towers into two opposite groups of three towers each and revealed red diagonals running in opposite directions in the groups (Figure 3):

T1:	Would you put these [towers with three colors] together? Are they similar in any?
Yonny:	No because there is no other similar [i.e. Opposite group] [puts the towers back into pairs of opposite towers]
T1:	Well, ok. All right. I see why these [YYY; BBB; RRR] would go together. Tanisha [student seating at the same table], you see his diagonals here? He's paired them up in these groups where he has these diagonals going down? [Turns back to Yonny] What about these here [towers with three colors]?
Yonny:	Those no… oooh. I think I got something. I think I got something. Oh. I am so smart. Like that [reorganizes the towers with 3 colors in two groups of three towers: BYR; BRY; RYB and RBY; YRB; YYB. A red diagonal running through each group]. Like that? Cause see… Oh shoot. See [explains how the groups are opposites] you got the two yellows and the [two] blues switch and you got these [red cubes] going in the same diagonal. You got these and blue right here and you got.
T1:	[Staring at the towers with three colors] Well, I see the diagonal here [in one of the groups] but I don't see [a diagonal there in the other group…]. Ooohhh it [the diagonal run] in the opposite way. Cool. Hum. Hum. Very cool.

Yonny presents a proof-by-cases argument. He built 9 groups/cases of towers with 3 towers each (data). Since he believes he has *all possible groups and all towers in each group* (warrant), he concludes, by the proof-by-cases argument (implicit backing), that there must be 27 towers three-tall in total (claim). The "opposites" strategy helps Yonny account for all cases/groups and the diagonals assures him that he has all towers in each case/group. However, Yonny does not initially apply the "opposite groups" and the "diagonal" strategies consistently across the arrangement. The towers with three colors are not (1) organized in opposite groups and (2) there are no diagonals running through them as is the case with other towers in the arrangement. The teacher's intervention helps Yonny addresses these challenges and it shows the importance of attending to how mathematical arguments may be represented in supporting argumentation (SCK). During the episode, the teacher pays close attention to the tower representation. This allows the teacher to see that towers with three colors are organized in opposite groups and there are no diagonals running through them as the other towers in the arrangement. The teacher then challenges Yonny organize the towers as a group (*Would you put these together?* Are they similar in any?) and to show diagonals running through the towers (*What about these here* [where are the diagonals]?). Yonny addresses these challenges successfully and is finally able to apply his proof-by-cases argument consistently to the entire tower arrangement. The previous episode showed that understanding students' arguments is key for supporting argumentation. This episode shows that attention to students' representations can help teachers identify and find ways to best support students' arguments (KCT).

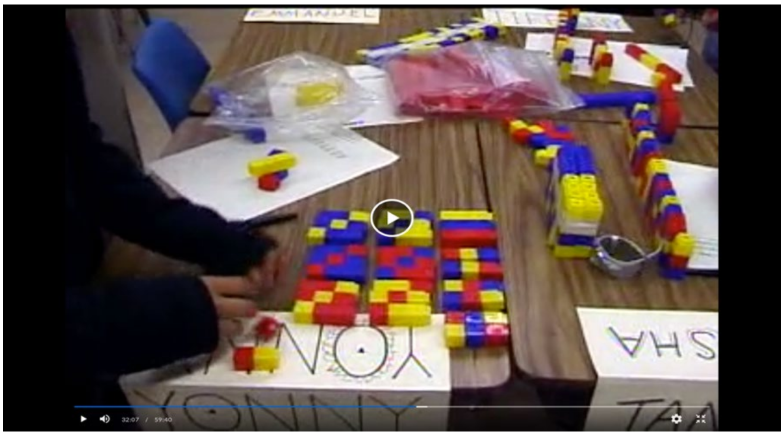

Figure 2. Yonny 's initial arrangement of towers for his solution to the 3-tal 3-colors problem

Figure 3. Yonny 's final arrangement of towers for his solution to the 3-tal 3-colors problem

4.5 Challenging arguments based on abductive reasoning

In IML sessions, there were several instances in which students justified their solutions by simply describing the models they were able to build to solve the tower problem. In the 4x4 argument above Gabe argues that there are in total sixteen towers four-tall when choosing from two colours because he built a model with four groups of four towers. In example below also involving the four-tall tower problem when choosing from two colors, James and Tanisha built four pairs of opposite towers and argued that there are eight towers in total because "4x2" equals eight:

T1: So, how can you prove to me that you have all of them?
James: I was thinking that *you have to multiply four by two because there are four cubes in a tower and there are two colors*. I mean, you have to multiply the height by the [number of] colors [in a tower]
Tanisha: And I said that's how you can find out how many towers we got. You can say *two* [opposite towers] *times four* [times] *equals eight* [towers].

James' and Tanisha's explanations simply describe the models they built. They built four pairs of opposite towers, which equals 4x2 or eight towers. When the teacher helped them see that they could build at least two more towers (YYYY and RRRR), bringing the total number of towers to ten, James said that the two extra towers "don't count' and Tanisha said "You will do five times two:"

T1: Now you agree that there are ten (towers). But *what happens to that two times four is eight and four times two is eight*, that mathematical thing that you were talking about?
Tanisha: [Reorganizes her towers into five pairs of opposite towers] I get the same. Because *you* still can do it my way, but it will just be five on the side and two. *You will do five times two*.
James: Now I am saying that these two [the extra towers], they are the same colors. They really don't count.

James and Tanisha continue to present explanations that describe the models that they built. James says that the two extra towers "don't count," which preserves his original explanation by applying it to the group of towers with two colors. Tanisha says "you still can do it my way…*You will do five times two,*" which is simply a way of counting the new set of five pairs of opposite towers that she was able to build with the addition of two extra towers. The students' emphasis on models that they were able to build suggests that the warrant supporting their arguments is empirical. For example, the 4x2 argument can be coded as follows using Toulmin's' scheme: There are be at least four pairs of opposite towers (data). Since no more towers *were found* despite trying (implicit empirical warrant), there must be only eight towers in total (claim). The empirical warrant is evident in Tanisha's response "you will do five times two" when she finds out that there can be two extra towers. She adjusts her response to the *new* set of towers that she has been able to find.

The 4xx and 4x2 episode shows the importance of knowing how to challenge abductive forms of reasoning (KCT). In abductive reasoning students present the *best or most plausible* explanation to support their claims and it can be the model they were able to build if it supports the claim that they want to make. In the 4x2 episode, four pairs of opposite towers (or 4x2) does equal to eight towers which the students believe to be the total number of towers because they could not find more towers. The challenge in countering arguments based on abductive reasoning is that it can be difficult to cause a cognitive disequilibrium in the students reasoning because the explanations presented are plausible or fit the argument that they are trying to make. However, the 4x4 episode above may suggest ways for challenging this type of reasoning. The teacher uses a suggestion from a researcher to ask the students to empirically investigate the validity of their prediction that there would be 3x3=9 towers three-tall when choosing from two colors based in their 4x4 model. The prediction does not hold which challenges the warrant in the 4x4 argument. This suggests that asking students to (1) empirically investigate the validity of general warrant that follow from their argument and/or (2) using counterexamples (the two-tall three-color tower problem) can help successfully challenge arguments based abductive forms of reasoning.

5 Discussion

This study examined the relationship between subject-related pedagogical knowledge and mathematical instruction using the Mathematical Knowledge for Teaching (MKT) framework. Specifically, the study examined how aspects of mathematical knowledge for teaching in the areas of SCK, KCS, and KCT that support teachers in promoting argumentation in mathematics classrooms. The results reveal several aspects including (1) knowledge of counterexamples, (2) a view of argumentation as a discursive process, (3) knowledge of (students') typical argumentative strategies, (4) representation of mathematical arguments and (5) knowledge how to challenge arguments based on abductive forms of reasoning. These aspects can help teachers elicit valid mathematical arguments from students in collective problem solving. The results offer important insights into teacher knowledge and practice of argumentation in mathematics classrooms.

There are several definitions of argumentation, which reflects different perspectives on argumentation and its function (Schwarz et Hershkowitz, 2010). Some perspectives emphasize cognitive and structural aspects and validation as the main function of argumentation. Other perspectives empathize social and discursive aspects and persuasion as the main function of the argumentation (See., van Eemeren et al, 1996; Krummheuer, 1995; Baker, 2003). In a study that examined how teachers select tasks to promote argumentation, Ayalon and Hershkowitz (2017) found that teachers emphasized socio-cultural aspects of argumentation including student-teacher interactions and collective processes of argumentation (where arguments are constructed and critiqued). Ayalon and Hershkowitz used this finding to recommend incorporating this dimension into current frameworks for examining the effectiveness of textbook tasks for promoting argumentation, which they argue tend to focus mainly on structural and cognitive aspects of argumentation. The results of this study provide further support for an emphasis on the socio-cultural view of argumentation, showing that it can help teachers support argumentation in mathematics classrooms by allowing them to engage students in mathematical discussions that help challenge invalid arguments.

Studies show that introducing representations or contexts that are familiar to students and using counter-examples are two of the least frequent instructional actions in mathematics classrooms (Cengiz et al, 2011). In this study, a teacher was able to identify elements of an emerging reasoning-by-cases argument by examining a student's tower representation (groups/cases of towers and a strategy for proving that all towers in a case were found) and then use it to challenge the student to apply the argument consistently across all cases and complete the argument. In another episode, the same teacher used a counterexample to successfully challenge an invalid warrant in a student's argument. This shows that students' representations and counterexamples can be important tools for supporting argumentation in mathematics classrooms and need to be emphasized more in mathematical instruction.

In many episodes in this study, supporting argumentation involved attending to and building on students' particular reasoning or arguments. A researcher suggested a counterexample that was used to challenge an invalid warrant based on her knowledge of how students reason when working on the Tower Problem. The attempt by teacher T5 to support a student, Yonny, in building an argument to support a solution to the three-color tower problem was constrained by the teacher's lack of familiarity with the use of the "diagonal" strategy in the problem to prove that all towers of a particular were found. In contrast, teacher T1 successfully helped the student develop a complete proof-by-cases argument after identifying aspects of the argument in the students' tower representation. These episodes highlight the importance of teachers' understanding of students' mathematical reasoning in supporting argumentation and suggest that the supporting argumentation is more likely to be effective when it builds on students' argumentative reasoning.

The results of this study show the challenges of countering arguments based on abductive forms of reasoning. In mathematics classrooms this type of reasoning is common and one way students often engage in such arguments is by offering explanations that simply describe the models that they built to solve a problem. The challenge in countering such arguments is the difficulty to cause cognitive disequilibrium in students' thinking because the models often support the solution. The results of this study suggest that teachers can challenge abductive types of arguments by inviting students to empirically investigate the validity of general warrant that support the particular argument through counterexamples. As students find out that the general warrant is not valid, they begin to question the validity of their argument.

The results may emphasize individual aspects of MKT knowledge in the areas of SCK, KCS, and KCT that help teachers support argumentation in mathematics classrooms. However, as some episodes suggest, a combination of aspects in the three areas is more likely to help teachers successfully support argumentation in mathematics classrooms. Being able to make sense of students' mathematical reasoning and arguments (KCS) can help teachers design appropriate interventions for supporting the students' development of valid arguments (KCT). However, making sense of students' mathematical reasoning and arguments may require teachers' understanding of and skills in argumentation as a mathematical process (SCK). The episode involving the counterexample helps illustrate the idea. The researcher suggested the counterexample (KCT) based on her knowledge from experience of students' reasoning when working on the tower problem (KCS) and also her understanding of the general warrant that was implicit in the student's 4x4 argument (SCK). The combination of aspects from the three knowledge categories helped successfully challenge the argument.

The results of this study have implications for practice and research. First, the results emphasize the importance of building on students' mathematical reasoning in supporting argumentation. This suggests that professional development programs need to particular attention to teachers' understanding of students' mathematical reasoning and argumentation if they are to prepare teachers to support the practice more effectively in mathematics classrooms. Second, teachers can build knowledge of students' mathematical reasoning and argumentation from experience. However, in this study a researcher came up with the counterexample used to challenge a student's argument based on the researcher's knowledge of how students' reason when engaging in the tower problem. This suggests that close collaboration between practitioners and researchers can help create important synergies for generating important knowledge of students' mathematical reasoning that can help teachers support argumentation in mathematics classrooms. Third, the results emphasize the importance of teachers paying more attention to students' mathematical representations and using more counter-examples in instruction. These are two of the least frequent instructional actions in mathematics classrooms. Yet, this study shows that they can help for support argumentation in classrooms. Finally, this study suggests that a potentially important area for research could be the extent to which professional development models such as the IML project involving a close collaboration between teachers and researchers in after-school settings can be successful in supporting teachers in building the knowledge they need to support argumentation and thoughtful mathematical activity in mathematics classrooms.

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