"Learning models": Utilising young students' algebraic thinking about equations

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The overarching aim of this article is to exemplify and analyse how some algebraic aspects of equations can be theoretically explored and reflected upon by young students in collaboration with their teacher. The article is based upon an empirical example from a case study in a grade 1 in a primary school. The chosen lesson is framed by the El'konin-Davydov curriculum (ED Curriculum) and learning activity theory in which the concept of a *learning model* is crucial. Of the 23 participating students, 12 were girls and 11 boys, approximately seven to eight years old. The analysis of data focuses on the use of learning models and reflective elaboration and discussions exploring algebraic structures of whole and parts. The findings indicate that it is possible to promote the youngest students' algebraic understanding of equations through the collective and reflective use of learning models, and we conclude that the students had opportunity to develop algebraic thinking about equations as a result of their participation in the learning activity.

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1 Introduction

Algebraic thinking is argued to be a key ability that children need to develop from an early age for their understanding of formal algebra in later years (Venenciano et al., 2020). In many countries, curricula and mathematical policy documents stipulates a teaching that promotes the youngest students' algebraic thinking (Cai & Knuth, 2011, see also Venenciano et al., 2020). Kieran et al. (2016, p. 1) explains that "[m]athematical relations, patterns, and arithmetical structures lie at the heart of early algebraic activity". At the beginning of 2000, four ways of addressing the issue of early algebra were defined as "(i) generalizing related to patterning activity, (ii) generalizing related to properties of operations and numerical structure, (iii) representing relationships among quantities, and (iv) introducing alphanumeric notation" (Kieran et al., 2016, p. 5). Representing relations among quantities as a teaching model refers to a curriculum developed by El'konin and Davydov (ED Curriculum) in which students' understanding of part-whole relationships is at the core (Schmittau, 2003, 2004, 2005). The ED Curriculum, with its roots in the cultural historical tradition of Vygotsky (1987), Leontiev (1978) and Galperin (1968), has been described within the research field of early algebra as a curriculum model with potential for developing young students' algebraic thinking (e.g. Dougherty, 2004; Kaput, 2008; Kieran et al., 2016; Carraher & Schliemann, 2014; Venenciano et al., 2020). However, Kieran et al. (2016) argue that more research and empirical examples of how such a curriculum can be realised in an everyday teaching setting are necessary. This article seeks to contribute with such an example based upon a case study depicting how some aspects of equations can be theoretically explored and reflected upon by young students. The students were invited to use a graphic model as a mediating tool (a learning model) in a teaching situation framed by the curriculum designed by El'konin and Davydov and its complementary learning activity theory (Davydov, 2008; Repkin, 2003; Schmittau, 2003, 2004).

In the two following Sections (1 and 2) we provide the framework for our aim and research questions. In Section 3, we provide a more detailed description of the learning activity and its central concepts. Methodology is presented in Section 4, followed by the result, divided into two parts and presented in Sections 5 and 6. The article ends with concluding remarks in Section 7.

1.1 Early algebra – realising a written curriculum

The field of early algebra is interested in the development of students' algebraic thinking and problem-solving abilities (Kieran, 2018; Radford, 2012, 2018; Radford & Barwell, 2016; Warren et al., 2016). This is sometimes related to teaching in which students are to be engaged in algebraic or theoretical work (Kieran et al., 2016). In developing these skills and abilities early, some researchers believe they are tackling a known problem with the commonly-used arithmetical foundation of algebra (Kaput, 2008; Lins & Kaput, 2004; Radford, 2006, 2010). As previously mentioned, the ED Curriculum is regarded as a promising alternative route when attempting to alter a teaching tradition that introduces students to algebra based on an arithmetic approach (Carraher & Schliemann, 2014; Kaput, 2008; Kieran et al., 2016).

1.1.1 Teaching for algebraic thinking

For very young students, the ED Curriculum comprises a series of deliberately sequenced problems of measurements that require students to expand known problem-solving methods and tools to develop their understanding at a theoretical level (Davydov, 1962, 2008; Schmittau, 2004, 2005; Sophian, 2007; Zuckerman, 2004, 2005). The idea of introducing numbers and mathematical operations through measurements is thus central. Schmittau (2004) argues, in line with Vygotsky, that relational analysis of quantities must precede the development of the concept of numbers. In a discussion of algebraic thinking, Schmittau and Morris (2004) claim that the ED Curriculum:

> [d]evelop[s] children's ability to think in a variety of ways that foster algebraic performance. First, it develops theoretical thinking, which according to Vygotsky comprises the essence of algebra. For example, the children develop a habit of searching out relationships among quantities across contextualized situations, and learn to solve an equation by attending to its underlying structure. … Their ability to interpret a letter as "any number" allows the teacher to introduce children to the kind of general argument that is the hallmark of algebraic justification and proof. (Schmittau & Morris 2004, p. 23)

Kieran (2004) provides the following definition of algebraic thinking:

> Algebraic thinking in the early grades involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting. (Kieran, 2004, p. 149)

Radford (2018, p. 8) highlights that a definition of algebraic thinking such Kieran's also needs to include a requirement that the students be able to treat "indeterminate quantities in an analytical manner." Thus, teaching aimed at developing algebraic thinking must support an analytical approach. Radford (2012, p. 119) argues that

> [w]ithin the theory of knowledge objectification, thinking is considered a relationship between the thinking subject and the cultural forms of thought in which the subject finds itself immersed. More precisely, thinking is a unity of a sensing subject and a historically and culturally constituted conceptual realm where things appear already bestowed with meaning and objectivity.

1.1.2 Teaching and development of theoretical thinking

The ED Curriculum draws theoretically on Vygotsky's idea that "teaching should take a leading role in relation to mental development" (Chaiklin, 2003, p. 169). From this perspective, the development of theoretical thinking requires a specially-organised practical activity – a learning activity in which students can reconstruct mathematical concepts, norms and values and thus learn to master culturally and historically developed theoretical ways of knowing. In mathematics, theoretical thinking is often exemplified by algebraic thinking (Krutetskii, 1976; Radford, 2021). Central to learning activity is the idea of ascending from the abstract to the concrete (Davydov, 2008). He claims that, if students first work theoretically on an object of knowledge to find embedded structural and general aspects of a concept as well as its conceptual relations, they can later find concrete instances of the theoretical knowledge. Dreyfus (2015, p. 117) argues:

> According to Davydov's 'method of ascent to the concrete,' abstraction starts from an initial, simple, undeveloped and vague first form, which often lacks consistency. The development of abstraction proceeds from analysis, at the initial stage of the abstraction, to synthesis. It ends with a more consistent and elaborated form. It does not proceed from concrete to abstract but from an undeveloped to a developed form.

However, the abstract structural and relational aspect of an object of knowledge is not available to the students through a teacher's direct instruction (Davydov, 2008; Schmittau, 2004), and thus, in realising a learning activity that enhances students algebraic thinking, a mediating tool – a learning model[1] (Gorbov & Chudinova, 2000) – that students can manipulate, change and examine when elaborating on and discerning the abstract content of an object of knowledge is necessary. Within learning activity theory, a learning model "fixates the universal relation of some holistic object, enabling its further analysis" (Davydov, 2008, p. 126).

2 Aim and research questions

Education realised through tool-mediated learning activities is thus a foundation of the ED Curriculum. However, realising this type of teaching places substantial demands on the teacher when, for example, designing tasks, initiating a problem situation or supporting student collective theoretical reflective work in the classroom (Kieran, et al., 2016). Even though there is research within the field of early algebra that seeks to develop teaching in line with the ED Curriculum (e.g., Dougherty, 2004; Schmittau, 2003; Sophian, 2007; see also H. Eriksson, 2021; I. Eriksson et al., 2021), we still do not have a substantial body of empirically-based knowledge about how to realise such teaching. Furthermore, there are few empirical examples of how teachers in a Western context can use learning models and collective reflections to support

[1] A learning model must not be understood as a mathematical model but a form of tool for visualising and elaborating core ideas.

student learning of algebraic ideas. There are even fewer empirical examples of how the ED Curriculum is realised in the experimental school – School No. 91 where it was designed (see below).

Given this, and based on a case study from School No. 91, our aim is to provide a concrete example of how a specifically-designed teaching can promote the youngest students' algebraic thinking. The aim is also to analyse which algebraic or structural aspects of equations are made available when the students and their teacher collaboratively uses a learning model as a mediated tool in a learning activity. The analysis is guided by the research questions (RQs):

- RQ1: What algebraic thinking on the relationship of the whole and its parts and the unknown in equations, can be discerned through a learning model in a lesson framed by principles of learning activity?
- RQ2: What, in student and teacher tool-mediated joint action, promotes exploration of the algebraic aspects of equations?

3 Learning activity

Learning activity theory must be understood in relation to specific theoretical content. For example, the ED Curriculum, as it is known in the West, is designed to realise learning activities in mathematics (Davydov, 2008; Dougherty, 2004; Schmittau, 2003, 2004; Schmittau & Morris, 2004; Sophian, 2007; Venenciano & Dougherty, 2014; Venenciano et al., 2020). The basis of the curriculum for the youngest students is measurement and units of measurement. This curriculum was developed experimentally at School No 91 in Moscow where, in 1958, El'konin and Davydov, in line with Vygotsky theoretical assumptions (Davydov, 2008), began their experimental research on the influence learning processes exert on student cognitive development.[2] Based on their experiment, El'konin, Davydov and their team proposed new content and new methods for learning and teaching mathematics and language in primary school.

[2] As aforementioned, a learning activity is theoretically built on Vygotsky's (1987) cultural historical theory and Leontiev's (1978) activity theory. Thus, Davydov and El'konin further developed the work begun by Gal'perin (1968) and formed two learning activity curriculums for reading and writing and mathematics, respectively.

3.1 Learning activity and learning models

In learning activity, the overarching goal is the development of student theoretical thinking and agency, that is, their capability to act and participate in a new and independent manner in different content-specific activities (Davydov, 2008; El'konin, 1999; Repkin, 2003; Rubtsov, 1991; Zuckerman, 2003). In order to invite the students into a learning activity, the teacher usually introduces a problem situation (Repkin, 2003) which must contain some abstract but central structural or theoretical aspects of specific content (an object of knowledge) that the students need to become conscious of. The teacher cannot merely present a problem and tell the students to solve it. In order to become involved in a learning activity, students must, through analysis of the situation, develop a motive for engaging in the activity, and then transform the problem into a learning task. The first step of student analytical work includes joint reflection on what previous knowledge and known tools (i.e. learning models) they can test (Davydov, 2008; Rubtsov, 2013; Zuckerman, 2004). Repkin (2003, p. 27) explains that students need "new modes of actions". Students must transform the initial problem situation into a learning task that implicitly leads them to discover new methods, or new tools, to solve the problem and the teacher encourages them to collectively reflect upon and defend and expand their solutions. The discussion does not end until the students have reached a conclusion they consider correct or plausible (Davydov, 2008; Schmittau, 2003, 2005; Sophian, 2007). However, the youngest students must learn how to work within a learning activity and are thus dependent on the teacher as a more knowledgeable other (Vygotsky, 1934/1963, 1987). A learning activity can make it possible for students to work within what Vygotsky (1934/1963) described as a zone of proximal development (ZPD).

To make it possible for students to explore "the abstract" of a specific object of knowledge, Davydov and his fellow researchers suggest that each learning activity must be realised with the help of learning models as visual mediating artefacts. The purpose of a learning model is to visualise the structural aspects of an object of knowledge and make it possible for the students to manipulate it during their analytical work. A learning model can take the form of a scheme, for example, depicted by line segments (such as in this article), or as a semiotic system, as for example, $A=B+C$. Davydov (2008, p. 95) explains that

> the structure of semiotic systems reproduces or copies the structure of the object. For example, a chemical formula has semiotic mediated function since

> its connections and sequence of elements convey the character of an actual chemical relation, the structure of a chemical compound. Of course, as in any other form of model, this reproduction is approximate, simplifying and schematizing the actual object.

A learning model may also be in a physical form, but in that case is mostly in combination with a symbolic model on the blackboard (Gorbov & Chudinova, 2000).

4 Data and methods of analysis

In this article, we present the results of a case study (Flyvbjerg, 2011; Yin, 2014) conducted in School No. 91 based on Inger Erikssо'n (Author I) visits to the school. Each visit, in 2013, 2016, 2017 and 2019, consisted of 5–9 hours of classroom observations in mathematics, a total of 27 lessons. Most of the lessons observed were conducted in the primary grades and several were taught by Ms Natalia Tabachnikova (Author II). As is characteristic of a case study, there were multiple sources of data (Merriam, 1998). All classroom observations were documented using video recordings, complemented with digital photos of student work and the blackboard text, and audio recordings or field notes from formal and informal follow-up discussions with the teachers, especially those who taught mathematics. During each lesson observed an interpreter provided in situ Russia-to-English, while some members of the local research team also attended the lessons and complemented with contextual comments. The main interest of a case study is what can be learned from a specific case or more precisely, how to gain new insight into local practice (Flyvbjerg, 2011). In this case, the interest was the function of the learning models in the students' collective exploration of structural aspects of equations. By choosing a single lesson, it is possible to make a more detailed analysis of the tool used and constituent actions. In order to understand what learning is made possible in a particular situation, for example during a single lesson, it is vital to become familiar with the daily teaching in a broader sense (Stake, 2005). The use of a learning model and the communicative actions in the lesson chosen is considered as representative for the lessons observed in total.

On the one hand, Author II, who taught the lesson chosen as the example for this article (see below), has an insider perspective on the practice analysed in this article. On the other hand, Author I, through her recurring visits to the school, has gained an outsider perspective. During the visits, Author I had several opportunities to discuss the principles of teaching, and the learning activity theory together with Author II, her colleagues and the local researchers. Further, the first author is familiar with the ED Curriculum through her own research (see e.g. Eriksson et al. 2021; Eriksson & Jansson, 2017; Eriksson et al., 2019; Wettergren et al., 2021). Milligan (2016) addresses the issue of researchers' positioning as an insider or an outsider and suggests that it is possible to develop a position of an inbetweener. We find this concept useful when describing our collaboration.

4.1 Data

The data for this article is from the observation of a typical ED Curriculum first grade lesson (7–8 years old). The 45-minute lesson was video-recorded by Author I in April 2017, when the first graders had been in school for approximately 7–8 months. Of the 23 participating students, 12 were girls and 11 boys. As a complement, some of the student worksheets were photo-documented. The video recordings, as the main source of data, were translated and transcribed into English by a researcher familiar with School No. 91 and learning activity theory. Author II reviewed the film and the first draft of the transcript. Finally, the transcript was jointly reviewed, and clarifications made, by Author I and Author II. The transcription captured all oral communication, complemented by gestures and intonation in situations where they provided meaning (Radford, 2010; Roth & Radford, 2011). The transcription was verbatim, speech neutral and organised dialogically (Linell, 1994). In the translation, nuances of the classroom interactions may have been missed in some cases (see Radford, 2010; Roth & Radford, 2011). Unfortunately, the sound quality was not always optimal, which may also have led to omissions, and the classroom atmosphere was not easy to capture in a transcript – at times the students were unable to sit still and wait to be called on. The atmosphere was intense and lively. To compensate for this, we repeatedly reviewed the transcript, the translation and the video/photo documentation of student actions, gestures and facial expressions.

Central to this lesson was the learning model depicting an algebraic structure of a whole and parts in the form of a line segment scheme |—|———| with which the students were familiar. The 'whole' was marked with a curved line on the upper or lower part of the model, and the parts were correspondingly marked with two shorter curved lines (illustrated below). In this lesson, the line segment model was presented in three drawings on the blackboard. Author II explained, in line with the ED Curriculum, that the overarching aim of the lesson was to stimulate student analytical and theoretical thinking, in this case in relation to the algebraic structure of equations.

4.2 Analysis

The results of the analytical work are divided into two sections (5 and 6). In order to establish an empirical foundation for analysis in relation to the two research questions, a narrative of the unfolding learning activity was constructed which comprised five identified sequences that captured the key events during the lesson. To identify the beginning and end of a sequence, focus was directed towards the teacher's communicative verbal and non-verbal actions that signalled such transitions, e.g. saying: "Look at the blackboard, please" [while she puts one forearm on top of the other—a known signal that students should be quiet]." Of the five identified sequences, three (first, third, and fourth) were the focus of this analysis. The second sequence was omitted due to silence while students worked individually. The fifth sequence was omitted mostly due to space constraints but also since this sequence repeated much of the action in the already selected sequences. Sequence 1 (approximately ten minutes) involved an introduction to the problem situation with the discussion prompt, *Three drawings: What is similar?* In Sequence 3 (approximately six minutes) the students wrote an equation for their problem: *Writing a programme for a calculator*. Finally, in Sequence 4 (approximately six minutes), the students reflected on the puzzling fact that there were three equations on the board but several problems presented by the students: *Three solutions but several problems*. The lesson concluded with Sequence 5, an additional task in which the teacher wrote $120 - x = 15$ on the board and asked the students to visualise this equation using the same line segment learning model from the previous task. An engaging discussion based on this new equation followed but is not included in this article. The narrative of the three chosen sequences is presented in Section 5. The second step in the analysis, presented in Section 6, aims to provide a more elaborated answer to the two research questions.

The analysis of the empirically based narrative was inspired by concepts related to learning activity (Davydov, 2008). From that perspective, human actions are understandable if it is possible to discern who does what (what are they doing), why (the goals of the actions), and with what tools (implied that all actions are tool-mediated)? In such an analysis, both oral and written speech, combined with the teacher and student intonations and gestures, provided analytical information when trying to capture what constitutive tool-mediated, goal-directed actions are occurring (Roth & Radford, 2011). In relation to research question 1, special attention was paid to which understanding of the constituent parts of an equation was made available through the joint tool-mediated actions. In relation to research question 2, special attention was placed on what in the tool-mediated joint actions enabled discernment of structural aspects.

4.3 Ethical considerations

Because School No. 91 is an experimental school, parents of students enrolled there provide consent for researchers and teachers to experiment with the curriculum, observe and videotape lessons and to study student learning. The researchers and teachers were permitted to use the videotapes and results of this study for only two purposes: scientific articles and teacher-training courses. In the transcripts, pseudonyms were used for the children and photographs were selected or retouched to reduce the possibility of identifying individual students. Individuals who know the students may, however, recognise them.

5 A narrative of the unfolding learning activity

The following sequences from the chosen lesson were described narratively and chronologically as the activity unfolded.

5.1 Sequence 1. The problem situation built into the three drawings: what is similar?

As the students enter the classroom, there are three drawings on the board (Figure 1), each based on the type of line segment model that the students are familiar with.

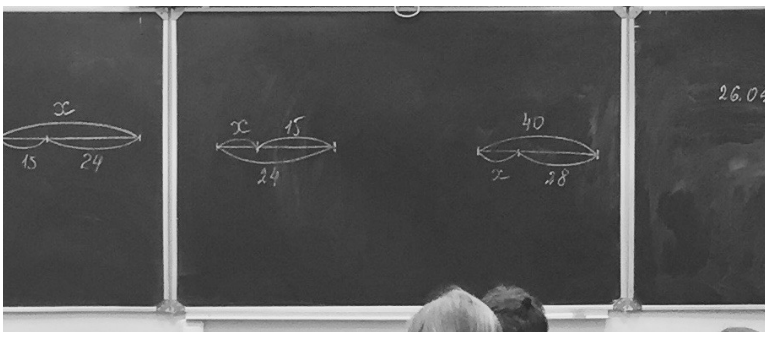

Figure 1. The three drawings on the board

The teacher asks the students to compare the three drawings and try to determine what they have in common. There is eager mumbling and several students raise their hands. The teacher asks a student to come forward and indicate her suggestions. The first thing that the student identifies is the x in each drawing and the teacher checks that everyone agrees on this assertion.

Excerpt 1.

Teacher:	Please look at the board. We see three drawings. What do you think these three drawings have in common? ...
Katya:	[Goes up to the blackboards and points] There is x.
Teacher:	Who agrees with Katya? Aha... All saw it. Wonderful! What else is common in all three drawings?
Students:	x
Olechka:	The three drawings have an x
Teacher:	Olechka, what is x?
Students:	Unknown.
Teacher:	The unknown. Dimitra, have you noticed anything else? Mila, what have you found?

The teacher asks for other similarities and various students come forward to show what similarities they have found, some mention the numbers in the drawings, others the structural aspects of a whole and two parts. The teacher then signals verbally and with gestures that there can or must be more similarities.

Excerpt 2.

Stepka:	Look, they are all similar! Here, they have a large part, a medium part and a small part [shows the first drawing]. Here is a large, medium and small [second drawing] and here too [third drawing].
Teacher:	Good. And you, what do you want to show us? ... Mila? You also found something they have in common ...
Dimitra:	I realised that in all three drawings the whole consists of two parts. In this they are similar.
Teacher:	So, children [turning to the class] do you understand what Dimitra means?
Dimitra:	I wanted to say that there is a whole and two parts in all the drawings.
Teacher:	Do you agree with that?
Children:	Yes, yes... but Stepka did say that...
Teacher:	I think Stepka said something else? Right Stepka?
Stepka:	I said all had a large, a medium and a small.
Teacher:	Yes. And Dimitra said that there is the whole, which consists of two parts. And all three are like that.

The teacher is obviously satisfied when Dimitra identifies the algebraic structure of the relationships between the whole and the parts in the different drawings. A structure can be expressed in various ways, as for example, $a+b=c$ or schematically as in the drawings. Using what can be seen as an imaginary playful format she then asks students to examine the structure in more detail: “You know, one boy from another class said one thing... He said that he thinks that two drawings are much more similar to each other than the third one. Which one is different?” During this sequence, several students come to the board simultaneously – all engage in explaining and arguing. Some students work together with the teacher, and some work in pairs (Figure 2).

Figure 2. The teacher signals ”I don’t understand.” Some of the students give an explanation and the teacher signals that she does not understand by lifting her shoulder and holding out her hands in a questioning gesture as she says: “Can that be?” or “Is this right?”

Excerpt 3.

Teacher: Raise your hands, please, who found the two drawings and sees their similarity and how the third is different? Michail. [Michail goes to the board and points to the first and second as similar]. So, these two [teacher points] are similar, and this one is completely different? [Teacher turns to the class and asks…] Yes? Who understood what Michail means? [the intonation in the teacher’s voice suggests that she doesn’t understand] ... Who can show and explain what these two drawings have in common? [A boy goes to the board and indicates ...]

A student: Here is [pointing] *x*, and here is *x*, here is 24, and here is 24.

Teacher: Michail. Did you mean this? [Michail nods]

Teacher: And who was thinking of something else?

On several occasions, the teacher involves the whole class by saying, for example, “Did he guess correctly?” and turns away from the students who have given the suggestion. The students are apparently used to participating in such collective discussions characterised by signals and gestures related to “agree” or “don’t agree”.

Excerpt 4.

Varya:	[Pointing to the first and second drawing] If we turn them over, it will be the same…
Teacher:	Have you also thought about these two? [pointing to the first and second drawing].
Varya:	Yes.
Teacher:	Varya agrees with you. And who had the other two [drawings] in mind? … What do you think? And you? And you? [the teacher addresses different children. A girl goes to the board and indicates the other two drawings – the second and the third].
A student:	These [pointing to the different parts – each line segment – of the second and third drawing] are similar, because here *x* is large [first drawing] and here [second and third] *x* is small.

From the video it is possible to note that many (if not all) students are involved: some are standing, while others have their hands raised and are eagerly calling out their suggestions. The atmosphere is intense and engaging.

Excerpt 5.

	[Several students are at the board, pointing and explaining]
Student:	These are similar because here *x* is large [first drawing] and here [second and third] *x* is small.
	[Another girl goes to the board and they both indicate]: Here *x* is big, and here, and here it is small.
	[A third girl comes up to them and says, while pointing to the second and third drawing…]
Olechka:	Here the *x* is a part [second and third drawing], and here the *x* is the whole [first drawing]
Katya:	[Pointing to some numbers] And here and here it is four.
Teacher:	Yes. Four is good. But it's more important to understand where *x* is the whole, and where it is the part.

The episode ends, and the teacher gives the students an assignment related to continuing their exploratory work with the learning models. The students are asked to copy one of the drawings on the board into their workbooks, but without telling anyone which one. When they have copied one of the drawings, the teacher asks everyone to write a story (see below) in relation to their chosen drawing – a problem in the form of a story using the whole and its parts in their drawing. When observing the students as they started to write their stories it was obvious that they were used to this type of work.

5.2 Sequence 3: A programme for a calculator – finding a concrete solution

After a while, the teacher invites several students to the front of the class to read their stories. In relation to the second drawing (see Figure 1), one of the students read:

> On his birthday Peter was presented with 15 new cars [Hot Wheels]. And now he has 24 Hot Wheels cars. How many Hot Wheels cars did Peter have before his birthday? ("Story" read by a student)

In relation to the third drawing (see Figure 1), another student read:

> There were 40 children on the school bus. 28 children are seven years old, the rest are eight. How many 8-year-old children are there on the school bus? (Story read by another student)

Next, the teacher asks the other students to guess which drawing the story is about. The duration of this process is approximately eight minutes, after which the teacher again calls for the students' attention. Under each drawing, she draws "x =" and asks the students to do the same under their chosen drawing.

Excerpt 6.

Teacher:	How will you find x? And, we need to make an action plan i.e., write an equation such as $x = a + b$ or $x = d + e$ or $x = 8 - 5$. If someone cannot calculate the result, that's not a big deal. We will choose a student who counts well, someone who will be a 'calculator' and they will count for everyone who has difficulties. For us it is important to just make an action programme for a calculator, all right?

With these instructions, student work takes a new direction. The teacher wants the students to write an equation for each of the three drawings that could be used to program an [imaginary] calculator, stressing that it is not necessary to figure out the answer, simply to write the 'program' in relation to the drawing they have chosen and the problem (story) they have written. In doing so, she uses what van Oers (2009) describes as a playful format to manage the fact that not all the students are able to solve the equation. The students immediately seem to grasp the imaginary calculator idea.

Excerpt 7.

A student	[calls out]: 24 + 15 = 39 [pointing to the middle drawing on the board].
Teacher:	You have already calculated! 24 + 15 = 39. Stand up, please, those who also have [points to the middle drawing]. There are so many of you! Two… no, four. Varya, please, what have you written?
Varya:	*x* is 15… [stops].
Teacher:	So, you think an unknown [*x*] plus 15 is 24?
Olechka:	No! [protesting] May I? [goes to board].
Teacher:	OK. Write.
Olechka:	[Writes and says] 24 - 15 = 9.
Teacher:	Aha. Look here, what Varya meant: how much should I add to 15 to get 24? But you did not write a programme for the calculator. How much should I add so that I get…? We should write a programme for the calculator to make it clear.
Mila:	[Now two girls are writing—almost on top of each other—on the board] $x = 24 - 15 = 9$ …

The teacher then asks for the students who have written an equation and a solution to the second and third drawing.

Excerpt 8.

Teacher:	Ok. Thank you. Now, who was solving this one [referring to the second drawing]? … Michail, come to the blackboard. Dina, you've been here already… ok, you may come and support Michail. [The boy writes behind the "*x* =" that was already on the board]: $x = 40 – 28$.
Teacher:	Let Mila continue.
Mila:	18 [writes "=18"].
Michail:	[Turns from the board to the teacher and says quietly while signalling with his finger] "I don't agree". I think it is 12.
Teacher:	You think it is 12… [looks inquiringly at the class]? [Children nod affirmatively and signal consent]. [Mila writes 12]
Teacher:	[Turning to Mila] Don't worry. You wrote the correct programme for the calculator. That is very important. And it will help us calculate. Thank you!
Teacher:	You have written very different programmes for the calculator with very different numbers and different answers. Here, two groups wrote minus in the programme for the calculator [points to two drawings on the right], others wrote plus [points to the drawing on the left]. What do you think? Why? Explain to me, please, when to subtract, and when to add.

At the end of the exercise, the teacher asks the students how it is possible that some of the programmes they have written for the calculator use minus and some use plus, which gets them to consider the algebraic relational structure of the whole and its parts and possible operational functions.

5.3 Sequence 4: Three solutions but several problems

The students are fully occupied with their assignment of writing programs for the calculator and explaining why subtraction or addition is required in some programmes, when the teacher gives them a question about how many problems they have created together and how many solutions are possible. At first, the students apparently struggle to understand what the teacher is asking for.

Excerpt 9.

Teacher:	... What do you think? How many solutions are written on the blackboard? [referring to the three equations with their respective solutions that the students have calculated on the board]. Show me with your fingers, how many. [Several students show three fingers in the air; the rest join them]. I can even count them. One, two, three... Yeah. And how many problems have we designed altogether [referring to the stories that the students had created earlier]? ...
Teacher:	If you want to answer, raise your hand. Why did this happen? Were there 12 (or 15) tasks and only three solutions? 12 tasks, then 12 solutions?
Students:	[In chorus] No!
Teacher:	Well ten at least...
A student:	[Approaching the board with hesitation] Because every task has one solution!
Teacher:	I don't understand. One solution? But here are three of them. Look: one, two, three. But there were 15 problems! How did this happen?
A student:	Because there are many people in the classroom!
Teacher:	There are many people in the classroom, that's why there are 15 problems, but solutions?
A student:	Everyone has their own answer!
Teacher:	Aha, so we have 15 answers?
Students:	No...
Teacher:	No... That is what I'm talking about. So, we have 15 problems and only three answers. Why?
Student:	Because everyone has his own answer, everyone wants to share his own knowledge!
Gavril:	Because we composed our problems for these three drawings. Therefore, there are three solutions...
Teacher:	I really liked what Gavril said. So, someone came up with a puzzle [a story] for this drawing [pointing to the first drawing], other children for the second [pointing to the middle drawing]. And which of you wrote a story for this drawing [points to the drawing on the right]? Therefore, we got only three solutions for many

different problems. [Finally, it seems the children realise how they got only three solutions for the 15 problems they compiled].

Teacher: Well done. Everyone did a good job.

In this episode, the teacher puts forward a problem that is difficult for the students to understand. "How can there be so many problems but only three solutions?" The teacher repeatedly encourages the students to provide an explanation for the *mysterious fact* that there are several problems or stories but only three solutions. First, when the teacher asks which students have created a story for the first drawing, and which for the second and third, it is possible for them (or most of them) to understand that one equation with its solution can match several concrete problems or stories.

6 Utilising young student algebraic thinking about equations

In this section, we analyse the narratively-depicted learning activity and its evolving in relation to the two research questions.

6.1 Algebraic thinking of the relationship of the whole and its parts and the unknown in equations

The first research question addresses the idea that algebraic thinking of the relationship of the whole and its parts and the unknown in equations can be discerned through a learning model in a lesson framed by learning activity.

In the basic line segment learning model exemplified in the three drawings (see Figure 1 above) the selected numbers and the placement of the *x* was important for making algebraic ideas related to equations possible to collectively discern and reflect upon. In one of the drawings, *x* represented the whole, and in the other two drawings it represented a part of the whole. How the teacher posed the questions and how she let the young students contribute different suggestions supported by gestures and language played a critical role when the students explored the three drawings. This made it possible for the them to discern that:

a) The symbol *x* can be used to symbolise something unknown that can be either a whole or one of the parts.
b) The problem embedded in the three drawings mathematically describes a relation between the whole and its parts (i.e. what the learning model with its line segments and the arches depicts).

In the following sequence, the students first secretly chose one of the drawings, wrote a story in relation to it, and read it aloud so that the other students could guess which model the story was written to describe. The students then wrote a "program for a calculator," that is, they wrote an equation that modelled the relationships. The teacher emphasised that it was not necessary to find the answer to 'the program' because the imaginary calculator could do that. However, we observed that all the students managed to find the answer to their equation. Using the learning model in this manner, the students had the opportunity to discern at least the following that:

c) A story or a visual representation (i.e. the learning model) can be 'translated' into an algebraic equation.
d) In an equation, the relational structure between a whole and its parts may vary, and the unknown can be either the whole or any of the parts (the exemplification of the learning model in the three drawings made it possible for the students to discern this).
e) A problem, when translated into a mathematical problem as a first step towards a solution can be formulated as an equation that will make it possible to determine the unknown – the value of x.

In the third sequence, when the students had written the programme for the calculator in their workbooks and on the board and calculated x in the three drawings, the teacher confronted them with a new problem. She asked the students how there could be so many problems (in the different stories created by the students) but only three solutions. In this situation the teacher addressed a topic that was apparently difficult for the students to figure out. The teacher, however, was persistent and posed the questions several times in various ways even though it seemed as if the students had more or less provided the same type of explanation. First, when the teacher called students to the board, and then in relation to each of the calculations the students demonstrated that the question could have an answer, some of the students wrote their problem for only one of the drawings. This contradictory question from the teacher made it possible for the students to discern that:

f) An expression or equation can represent different contextual problems or situations.

This can be described as an emerging algebraic thinking of the generality of equations and thus a first step in being able to ascend from the abstract to the concrete.

6.2 The student and teacher tool-mediated joint actions

The second research question addresses what in the student and teacher tool-mediated joint actions promotes exploration of the algebraic aspects of equations.

The three original drawings on the board with the question "How are all these three drawings similar?" can be considered the introduction to the first problem situation. The students were to identify and analyse different relational aspects built into the problem situation with the help of the learning model used in the three drawings. The analytical or theoretical work was conducted jointly, and the students apparently challenged each other to find more similarities.

g) Student theoretical work was collectively realised by those at the board and the others who remained at their desks through (previously agreed-on) hand signals of agreement or disagreement. Several students also verbally expressed whether they agreed or not. This joint labour, as Radford (2018) describes it, made it possible for the students to both see and hear others' suggestions and explanations while simultaneously expressing their own understanding. This can be described as a collective reflection (Zuckerman, 2004).
h) While students at the board used the learning model and its components to make their thinking and suggestions accessible to others assessment the teacher often acted *as if* she did not really understand what the students were trying to say and mostly signalled this with gestures and by asking other students to explain. This promoted the students to elaborate the content further.
i) The teacher's 'unwillingness' to understand, combined with the way the three drawings were designed (based on the line segment learning model), allowed the students to elaborate on the algebraic ideas of the whole and its parts, and x symbolising the unknown. Understanding that the unknown symbolised by an x can be any part of an equation, the whole or one of the parts.

To summarise, this can be described in terms of materialising student collective algebraic thinking (Radford, 2006, Venenciano et al., 2020). The learning models and the problem situation, combined with the communication prompted by the teacher, made it possible for the students to reflectively take the others' position while simultaneously better understanding their own ideas (Zuckerman, 2003, 2004).

7 Concluding remarks – teaching that enables and enhances algebraic thinking

The learning activity realised in second author's classroom can be described as analytical and reflective. The structure and use of the learning model combined with the teacher's prompts and her ability to take advantage of student answers and questions, created opportunities for the students to analytically reflect upon others' suggestions and explanations. That is, the students could use each other's thinking (visualised with the help of the learning model) to further their own thinking (Zuckerman, 2003, 2004). Furthermore, the opportunities for the students to act and express their ideas and to have these elaborated by others appeared to promote the development of their agency (Davydov et al., 2003).

Following students' joint actions from Sequence 1 through Sequence 3, there are indications that they increasingly expressed themselves analytically and mathematically (Radford, 2018). First, the three relational aspects embedded in the drawings were not discerned by the students. They talked about smaller and bigger parts, but not of how they were related to each other. Thus, the students did not initially reflect upon what the whole was and what the parts were and what in that structure was known and what was not. Second, the analytical work that was required of the students when asked to create a word problem and an equation for one of the drawings made a mathematically-relevant understanding possible. Given these aspects, because of the student participation in the learning activity, it was possible for them to develop complex relational thinking regarding, for example, possible structures of equations, the unknown and the relationship between equations and contextual situations. Regarding quantities – as mentioned in the introduction, Radford (2018) addresses the need to consider student analytical work as an indicator of algebraic thinking. If a student merely guesses or uses a trial-and-error strategy and produces relevant answers, this does not count as algebraic thinking. Thus, it seems plausible that the students had opportunity to develop complex relational thinking because of their participation in the learning activity.

In a learning activity such as that that evolved during this lesson, several aspects must occur simultaneously. Because the object of knowledge embedded in the problem situation and the learning model may at any moment, be at risk the teacher and student co-actions are significantly important. In particular, the teacher needs to consciously address individual student suggestions and explanations, making them available for the other students to continued exploration.

This study may be considered as limited in that only one lesson was analysed. However, the lesson chosen out of 27 observed lessons is representative in relation to the aim of teaching and the use of learning model as a mediating tool for students' problem-solving theoretical work (Larsson, 2009). Thus, we hope that our analysis can provide some indication that it is possible to illuminate algebraic ideas through the collective and reflective use of learning models. This may be regarded as a way to allow for complex relational thinking (Davydov, 2008) to take a materialised form that others are able to reflect upon (Radford, 2018, 2021; see also H. Eriksson & I. Eriksson, 2020; Eriksson et al., 2019).

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