Student teachers' knowledge of students' difficulties with the concept of function

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An important part of the mathematics syllabuses at the secondary school level in most countries is the concept of function. However, secondary school students often experience difficulties with this concept. These difficulties are well-known in the research literature. The study applies the mathematical knowledge for teaching (MKT) framework, including the category knowledge of content and students (KCS). Teachers' ability to anticipate students' difficulties is one aspect of KCS. The aim of this study is to investigate secondary mathematics student teachers' KCS regarding the concept of function. Ten mathematics student teachers participating in a Supplementary Teacher Education Program answered a questionnaire about fictive secondary school students' various difficulties with the concept of function. Follow-up interviews were conducted with four of the respondents. Compared to the findings of previous research on students' difficulties with the concept of function, the respondents in the study sometimes provide reasonable suggestions about the sources of students' difficulties. Some of the respondents demonstrate an aspect of KCS when they suggest that students can reason that a function must be defined by one algebraic expression only, and that students only know about continuous functions. However, no respondent suggests that one source of students' difficulties with a constant function with an implicit domain is the missing domain. In addition, some respondents take for granted that students can interpret the algebraic representation of a piecewise-defined function and translate it into a graph.

Keywords: The concept of function, teacher knowledge, student teacher, mathematical knowledge for teaching (MKT), knowledge of content and students (KCS)

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1 Introduction

The concept of function is an important part of mathematics (Freudenthal, 1983), and of mathematics syllabuses at the secondary school level in most countries (National Council of Teachers of Mathematics, 2017; Swedish National Agency for Education, 2012). However, this concept is difficult to master. Secondary school students often experience difficulties with, for example, constant functions, piecewise-defined functions, and with the one-valuedness property of a function (Clement, 2001; Hatisaru & Erbas, 2017; Tall & Bakar, 1992; Vinner & Dreyfus, 1989). Teachers' knowledge about students' misconceptions, and how to overcome them, is one aspect of *pedagogical content knowledge* (PCK) (Shulman, 1986). Teachers' PCK has a positive effect on students' learning gains (Baumert et al., 2010).

Ball, Thames and Phelps (2008) conceive *mathematical knowledge for teaching* (MKT) as a development of PCK. There is a positive correlation between teachers' MKT and student achievement gains (Hill, Schilling & Ball, 2004; Hill, Rowan & Ball, 2005). Teachers' ability to anticipate and resolve students' errors and misconceptions regarding the concept of function, ability to interpret students' incomplete reasoning, and to anticipate what tasks students will experience as difficult are aspects of *knowledge of content and students* (KCS) which in turn is part of MKT (Nyikahadzoyi, 2015). This knowledge influences the teacher's decision on how to respond to students' questions (Even & Tirosh, 1995). Hence, student teachers' need to develop their level of KCS in order to enhance students' understanding of the concept of function. Therefore, it is valuable to investigate student teachers' knowledge of the sources of secondary students' difficulties with the concept of function.

1.1 Research question

What *knowledge of content and students* (KCS) do the participating secondary mathematics student teachers demonstrate regarding the concept of function? In particular, suggestions about students' difficulties in recognizing constant functions and piecewise-defined functions, difficulties regarding the one-valuedness property of a function, difficulties related to the various representations, and students use of prototype examples are considered.

2 Background

2.1 The concept of function

In this study, we define the concept of function as follows: Let D and S be two nonempty subsets of the real numbers. A function from D to S is a rule that assigns *exactly one* number in S to each number in D. The last part of the definition is referred to as the one-valuedness property of a function. The two sets D and S are called domain and codomain.

Now we give a summary of how this concept is defined in Swedish upper secondary school mathematics textbooks. The concept of variable is usually defined as a letter in an algebraic expression that can assume different values. Two of the most frequently used Swedish mathematics textbooks for upper secondary school (Sönnerhed, 2021) define a function as a relationship between two variables that satisfy the one-valuedness property:

> If the relationship between two variables x and y is such that each x-value, according to some rule, gives a unique y-value, we say that y is a function of x. (Alfredsson, Bråting, Erixon, & Heikne, 2011, p. 288).

> A function is a relationship or a dependency between two variables. It is said that y is a function of x, if for each value of x there is a unique value of y. (Szabo, Larson, Viklund, Dufåker, & Marklund, 2011, p. 162).

In this study, x and y are referred to as the independent and the dependent variable, respectively. The textbooks define the domain of a function as all the values that the independent variable can assume. The codomain of a function is defined as all values of the function when the independent variable is selected from the domain. The domain of a function is most often implicit, that is, not explicitly stated. When the domain is implicit, it is a convention to assume that it is the largest set of real numbers for which the rule of the function makes sense (Adams, 1995).

A function can be represented in different ways, for example, with an algebraic expression, a table of values, a graph, a verbal description (Chang, Cromley & Tran, 2016), or an arrow diagram (Markovits, Eylon, & Bruckheimer, 1986).

2.2 Students' difficulties with functions

The research question concerns student teachers' knowledge of secondary students' difficulties with the concept of function; therefore, a review of previous research literature is presented on such difficulties. In this study, we have chosen five difficulties with the concept of function which are well known from the previous research literature. We have investigated if student teachers know about the sources of these difficulties by requesting their suggestions about fictive students' possible reasoning about the concept of function.

The first example of such difficulties is the constant function with an implicit domain. Students can have difficulties recognizing constant functions with an implicit domain because they expect an independent variable in the algebraic representation of a function; when there are no independent variables in the algebraic expression, they do not regard it as a function (Hatisaru & Erbas, 2017; Tall, 1992; Tall & Bakar, 1992; Vinner & Dreyfus, 1989). Tall and Bakar (1992) ask secondary school students if a horizontal line in a coordinate system represents a function. Almost 50 % of the responses state that it does not represent a function, and about 70 % of the students answer that the corresponding algebraic expression does not represent a function. Thus, the choice of representation of the constant function was critical for the students' misconceptions. Also, secondary mathematics student teachers can have difficulties recognizing constant functions (Viirman, Attorps, & Tossavainen, 2010).

The second difficulty we have chosen to include in this study is that of piecewise-defined functions; that is, functions defined by different expressions on different subdomains. Such functions present difficulties for secondary school students (Hatisaru & Erbas, 2017; Tall, 1992; Tall & Bakar, 1992; Vinner & Dreyfus, 1989). In a survey study, Vinner and Dreyfus (1989) investigate college students' *concept images*, i.e. "the total cognitive structure that is associated with the concept" (Tall & Vinner, 1981, p. 152). The students were supposed to identify the graphs of two piecewise-defined functions — one continuous and one discontinuous — in a questionnaire. Some of the students propose that a function, which is represented with a graph, must be continuous and that it cannot be defined by different expressions on different subdomains (Vinner & Dreyfus, 1989).

The third difficulty we have included in this study is that of the one-valuedness property of a function. This property presents difficulties for secondary school students (Tall, 1992; Tall & Bakar, 1992; Vinner & Dreyfus, 1989), and also for secondary mathematics student teachers (Viirman et al., 2010). About two-thirds of the secondary students in the study of Tall and Bakar (1992) propose that a circle is the graph of a function. The authors' explanation for this misconception is that students' reasoning about functions rely on properties of familiar examples, such as circles or polynomials, and this familiarity evokes the concept of function. Thus, students do not check the one-valuedness property.

The fourth difficulty we have included in this study is that of students' ability to use multiple representations and the ability to translate between representations, for example, an algebraic expression or a graph. This ability develops a better conceptual understanding (Chang et al., 2016; Even, 1998). However, students encounter difficulties when translating between representations of functions (Bossé, Adu-Gyamfi, & Cheetham, 2011; Hitt, 1998).

The fifth difficulty we have included in this study is that of students use of *prototype examples*. Schwarz and Hershkowitz (1999) assert that when students try to understand a concept, some examples are more central in understanding the concept than others. Students use these *prototype examples* to decide whether other examples can be considered to belong to the given concept. Some students use linear and quadratic functions as prototype examples instead of using the definition of the concept (ibid.). In a survey study, Markovits et al. (1986) include a task in a questionnaire with two given points in a coordinate system and instruct the secondary school students to draw graphs of a function that passes through the two points. About half the students only drew the straight line, which is determined by the two points. The authors conclude that "there was an excessive adherence to linearity" (p. 24), and that this may have been caused by the time spent studying linear functions in algebra teaching.

In connection with solving equations, x is often called a variable; Kilhamn (2014) emphasizes that x should be named unknown instead of variable in this context.

2.3 The MKT framework

Teachers need knowledge of the sources of students' difficulties with the concept of function in order to improve students' achievements (Hatisaru & Erbas, 2017; Tasdan & Koyunkaya, 2017). Therefore, a literature review on teacher knowledge is presented.

The prevailing conceptions of teaching among policymakers and teacher educators who were contemporaries of Lee Shulman were that general pedagogical knowledge and some content knowledge was sufficient for teaching. Shulman (1986) argued that this ignored the complexities of teaching; instead, he emphasized the role of content in teaching. Hence, Shulman (1986) proposed a content-specific teacher knowledge referred to as *content knowledge for teaching,* including *subject matter content knowledge, curricular knowledge* and *pedagogical content knowledge* (PCK).

PCK includes the most useful forms of representing the content in a way that make it comprehensible to students. PCK also includes teachers' understanding of students' preconceptions of various topics. If these preconceptions are misconceptions teachers need knowledge of how to identify and overcome them (ibid.). PCK is the most influential of these three categories of knowledge (Ball et al., 2008).

Several well-proven extensions of Shulman's framework *content knowledge for teaching* have been developed with the aim of measuring teachers' knowledge for teaching mathematics (Kaarstein, 2014); for example, *professional knowledge of secondary school mathematics teachers* (Baumert et al., 2010), *teacher education* *and development study in mathematics* (Tatto et al., 2008), and *mathematical knowledge for teaching* (MKT) (Ball et al., 2008).

Ball et al. (2008) conceive MKT as a refinement of two of Shulman's (1986) categories of knowledge: *subject matter content knowledge* and PCK. It consists of six categories of knowledge: C*ommon content knowledge* (CCK) is mathematical knowledge not unique to teaching; it is needed by teachers and non-teachers. S*pecialized content knowledge* (SCK) is "the mathematical knowledge and skill unique to teaching, for example, finding an example to make a specific mathematical point" (Ball et al., 2008, p. 400). *Horizon knowledge* is an awareness of the relations between the mathematical topics included in the curriculum.

In *knowledge of content and students* (KCS), knowledge of common student conceptions and misconceptions is combined with knowledge of the content; for example, teachers need to predict whether the students will find the content easy or difficult and they also need to interpret students' incomplete reasoning. A teacher who has seen a misconception of a certain concept before in her teaching is able to recognize it without effort when she encounters the misconception again. In *knowledge of content and teaching* (KCT), knowing about teaching is combined with knowledge of the content, for example, how to sequence the content in the teaching. *Knowledge of curriculum* is self-explanatory. The framework MKT has been derived primarily from elementary school teachers' practices. A summary of the six categories of *mathematical knowledge for teaching* (MKT) is given in Table 1 below.

Table 1. A summary of the components of *Mathematical knowledge for teaching* (Ball et al., 2008).

	Categories of knowledge	**Description**
Subject matter content knowledge	Common content knowledge (CCK)	mathematical knowledge not unique to teaching
	Specialized content knowledge (SCK)	mathematical knowledge and skill unique to teaching
	Horizon knowledge	awareness of the relations between the mathematical topics included in the curriculum.
Pedagogical content knowledge	Knowledge of content and students (KCS)	knowledge of common student conceptions and misconceptions combined with knowledge of the content
	Knowledge of content and teaching (KCT)	knowledge of teaching combined with knowledge of the content
	Knowledge of curriculum	knowledge of curriculum

Nyikahadzoyi (2015) provides two examples of students' difficulties with the concept of function: To translate between representations and to interpret symbols related to functions. Teachers should be aware of the sources of the misconceptions associated with the use of certain representations, such as the function box that can lead to the misconception that all functions can be expressed with a formula. Identifying functions with the algebraic representation only can cause students to perceive functions as rules with a certain regularity, where a change in the independent variable causes a change in the dependent variable. One consequence may be that some students do not recognize constant functions with an implicit domain.

2.4 Teacher knowledge

Even and Tirosh (1995) examine 162 prospective secondary mathematics teachers' knowledge of students' conceptions, and also the sources of students' misconceptions related to functions. The authors use an open-ended questionnaire with fictive students' erroneous answers and misunderstandings of the concept of function. The prospective teachers were supposed to respond to the fictive students' erroneous answers. The authors conclude that several of the prospective teachers did not understand the sources of the students' misconceptions related to functions.

Hatisaru and Erbas (2017) investigate two secondary school teachers' levels of KCS regarding the concept of function with the use of two tasks in a test, where the teachers are asked to provide suggestions on the sources of a fictive student's difficulties regarding the concept of function. One of these tasks concerns a fictive student's difficulty with six different representations of six different functions: an arrow diagram representing a non-injective function, the graph of a discontinuous function, the algebraic representation of a piecewise-defined function, a verbal description of a function, a constant function with an implicit domain, and a set of ordered pairs of numbers. The authors' other task uses two given points in a coordinate system, where the fictive students were supposed to draw graphs of a function that pass through the two given points. This last task is also used by Markovits et al. (1986) with the purpose of investigating whether students use linear functions as *prototype examples* of functions. One of the two teachers demonstrate an aspect of KCS when she says that "the student may have thought that a function should be given by *one* rule only" about the algebraic representation of the piecewise-defined function in the test, and also when she says "it does not involve x", about the constant function with an implicit domain in the teacher test (ibid, p. 13).

Tasdan and Koyunkaya (2017) investigate prospective secondary mathematics teachers' MKT regarding the concept of function. The authors' findings indicate that the three participating prospective teachers had limited knowledge of how to anticipate what students will find difficult, and how to interpret students' incomplete thinking about the concept of function.

3 Method

3.1 Instruments

A questionnaire and follow-up interviews were used to collect data. Combining these two instruments to collect data about teachers' knowledge for teaching mathematics is a well-tried method (e.g. Even & Tirosh, 1995; Hatisaru & Erbas, 2017). A questionnaire was designed, including open-ended tasks which are about fictive secondary school students who have various difficulties with the concept of function. The tasks can be found in the Results chapter below. The intention of the questionnaire was to investigate secondary student teachers' level of KCS regarding the concept of function by requesting their suggestions about fictive students' possible reasoning. During the design of the questionnaire, inspiration for Task 1 and Task 2 in this study was taken from the tasks concerning students' difficulties with constant functions and piecewise-defined functions in the studies of Tall and Bakar (1992) and Vinner and Dreyfus (1989). In addition, inspiration for Task 6 was taken from a task in Hatisaru and Erbas (2017) with an arrow diagram representing a non-injective function, and for Task 8 from the task in Hatisaru and Erbas (2017) with two given points in a coordinate system, where fictive students were supposed to draw graphs of a function that pass through the two points.

A semi-structured interview (Bryman, 2012) was used with the purpose to further investigate the student teachers' knowledge of the sources of students' difficulties with the concept of function. During the interviews, follow-up questions based on the respondents' written suggestions in their questionnaires were asked task-by-task. The individual interviews were held in a seminar room and lasted approximately an hour each. All interviews were recorded using a digital audio recorder and transcribed verbatim.

3.2 Participants

The participants of this study were in the middle of a one-year teacher education program at the University of Gothenburg, referred to as the *Supplementary Teacher Education Program* (*Kompletterande pedagogisk utbildning*, KPU) *with increased study rate*. The program was designed for university students who have already completed a bachelor's degree in biology, physics, chemistry, mathematics, or technology, and wanted to become certified teachers in Swedish secondary education. During the clinical training (VFU), they talked about their teaching experiences at the university and at school. These talks took place in the form of dialogue seminars. During the training, their own recorded lessons were used as a basis for discussion and analysis. (Kompletterande pedagogisk utbildning, Ma/Nv/Tk, förhöjd studietakt, 2018).

Thirteen student teachers participated in a seminar entitled "An Introduction to Mathematics Education" which was a part of their teacher education. This was the very first time in their teacher education that they had formal training in mathematics education at the university. The questionnaire was distributed to the student teachers after the seminar. Ten of the thirteen student teachers answered it. They received the following pseudonyms: Bo, Dan, Eric, Fredrik, John, Patrick, Rickard, Sven, Tom, and Viktor. Four of them gave consent to be interviewed: Dan, John, Patrick, and Sven. Six of the ten respondents had only a short teaching experience, and the other four had none, before they were enrolled in the program. During their first semester of the *Supplementary Teacher Education Program,* they gained experience in teaching in clinical training (VFU) at about half time.

The participating student teachers had strong *subject matter knowledge* of the concept of function when they were enrolled in a teacher education program. This was assessed with the aid of a questionnaire concerning fictive secondary school students' erroneous statements about some examples of functions (NN, 2017, p. 93-95). This questionnaire was distributed to the participating student teachers the very first day of their teacher training at the university. At the same time, some background information was collected from the participating teacher students. A summary of the participating student teachers' ECTS points in mathematics, academic degree and teaching experiences is presented in Table 2 below.

Table 2. The participating student teachers' ECTS points in mathematics, academic degree and teaching experience.

Student teacher	ECTS points in mathematics	Academic degree	Teaching experience (month)
Bo	120	Master of Science	No
Dan	90	Master of Science	1-3
Eric	60	Master of Engineering	1-3
Fredrik	60	Master of Engineering	one semester
John	60	Master of Engineering	No
Patrick	220	Master of Science	1-3
Rickard	60	Master of Engineering	No
Sven	45	Master of Engineering	1-3
Tom	60	Master of Engineering	No
Viktor	60	Master of Engineering	one semester

3.3 Method of analysis

Qualitative content analysis is a research method for the analysis of texts aiming at an objective, systematic and replicable account of the content of the text. The method is applicable to different forms of information, for example, transcripts of semi-structured interviews (Bryman, 2012).

The respondents' suggestions, on how the fictive students in the questionnaire may have reasoned, were analysed task-by-task. After reading the questionnaires, quotes were identified where it was clear that the respondents suggest how the fictive students in the tasks may have reasoned about the sources of students' difficulties with the concept of function. Similar quotes were grouped and categories were formulated task-by-task. Hence, the categories emerged during data analysis through qualitative content analysis of data, that is, the categories were not given in advance in the questionnaire. Then the categorizations were validated by two of my supervisors.

The analysis of the questionnaires and the transcripts of the interviews were focused on the respondents' suggestions concerning students' difficulties in recognizing constant functions and piecewise-defined functions, difficulties regarding the one-valuedness property of a function, difficulties related to the various representations, and students use of prototype examples. Two tasks (Task 3 and Task 7) in the questionnaire were excluded from the analysis because they were not considered to contribute to answering the research questions.

Because the questions in the questionnaire were inspired by previous studies, the validity of the present study was improved. Although there were few respondents in the study, combining questionnaire and follow-up interviews to collect data about student teachers' knowledge for teaching mathematics improved the reliability of the study.

4 Results

The results are presented task-by-task. The respondents' suggestions on how the fictive secondary students in the questionnaire may have reasoned are presented in different categories. The various categories are illustrated with one representative quotation of the student teachers' responses. The numbers in parentheses show the number of suggestions in the respective category. In addition, interviews with Dan, John, Patrick, and Sven are presented. A summary of the respondents' suggestions on the tasks in the questionnaire is attached in Appendix A.

Task 1

> To a question from the teacher, whether $y = 4$ is a function, Ahmad answers no.
> How can Ahmad have reasoned? Please give several possible explanations!

A: Ahmad expects an independent variable in an algebraic expression representing a function. (8)

> He would like to see a dependency on a variable. John

Eight respondents suggest that Ahmad expects an independent variable in the expression $y = 4$. Two of the respondents did not recognize this difficulty.

B: Ahmad reasons that the expression $y = 4$ is an equation with one unknown. (5)

> The student thinks that y is an unknown number in an equation. Patrick

Five respondents suggest that Ahmad may have perceived the expression $y = 4$ as an equation with one unknown instead of a function.

All the four student teachers who were interviewed suggest that Ahmad expects an independent variable in the expression $y = 4$. Patrick suggests that the teacher can write $y = f(x) = 4$ instead of $y = 4$, where f denotes the function. In this way, it becomes clearer that a function is represented, and Ahmad's misconception can be avoided. However, the teacher raises the level of difficulty if she writes $f(x) = 4$ instead of $y = 4$ because students may have difficulty with parentheses in connection with algebraic expressions, according to Patrick.

Dan, Patrick, and Sven also suggest that Ahmad perceives the expression as an equation with one unknown, and that one should distinguish between a variable in connection with functions and an unknown in connection with an equation. Patrick puts the reasoning slightly forward when he suggests that there is a causal relationship between Ahmad lacking a dependence between two variables, and that he interprets the expression as an equation with one unknown instead of a function.

Sven suggests an exercise he calls "guess my rule". He suggests using, among other examples, a rule that always gives the value four[1]. This should give a rewarding discussion among the students about what a function is, according to Sven.

Task 2

To a question from the teacher, whether $y = \begin{cases} x - 3, if\ x \leq 0 \\ x + 3, if\ x > 0 \end{cases}$ is a function, Benjamin responds no. How can Benjamin have reasoned?

A: Benjamin may have difficulty interpreting this representation. (4)

> Benjamin can have reasoned that a function must be represented with one expression only. Patrick

Four respondents suggest that Benjamin may have difficulty interpreting the algebraic representation of this piecewise-defined function.

B: Benjamin reasons that a function must be continuous. (8)

> There is a jump in the curve. Tom

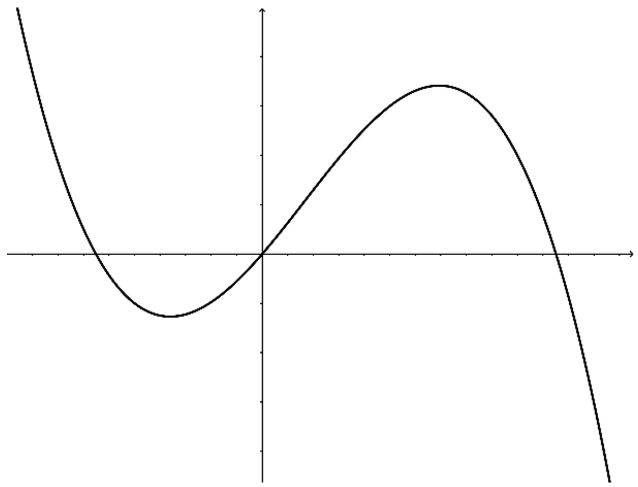
[1]"Guess my rule" is a game in which one student (or the teacher) gives examples of some unknown rule, and the other students try to discover the rule based on the given examples.

Eight respondents suggest that Benjamin can interpret the algebraic representation of the piecewise-defined function, translate it to a graph, discern a jump in the graph and conclude that it is not continuous.

During the interviews, Dan, Patrick, and Sven mention that the difficulty with seeing that a function is represented is that it is written on two lines with two algebraic expressions, using "if". Patrick also suggests that Benjamin can believe that this represents a system of linear equations with two equations and two unknowns. Another suggestion from Patrick is that Benjamin can translate this algebraic representation to a graph, and he concludes that it is not continuous *since there is a jump in the graph*. Because Benjamin takes for granted that a function must be continuous, he draws the erroneous conclusion that this does not represent a function.

Task 4

To a question from the teacher, whether the graph below represents a function, Daniel answers yes. How can Daniel have reasoned?



A: Daniel reasons that the curve looks like the graph of a function; therefore, it represents a function. (6)

> It's similar to the graphs of functions that you work with; for example, a third-degree function. Viktor

Six of the respondents suggest that Daniel reasons that the curve looks like the graph of a function because he recognizes the curve.

B: Daniel reasons that the curve is one-valued and therefore concludes that it represents a function. (7)

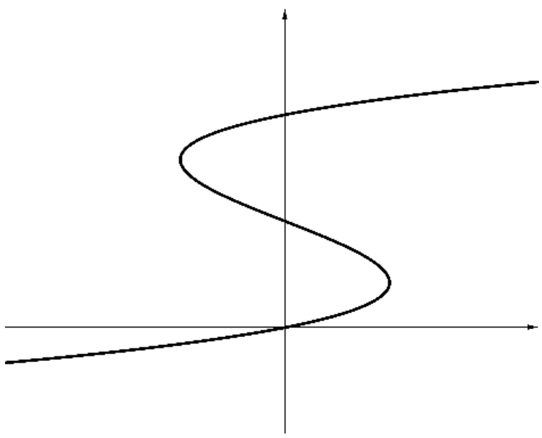
> There is one and only one value of the function to every value of x in the domain. Dan

Seven respondents suggest that Daniel may have reached his conclusion using the one-valuedness property of a function.

All student teachers who were interviewed suggest that Daniel recognizes the curve in Task 4 as a third-degree curve; therefore, he does not need to consider the definition of the concept. On the other hand, John, Dan and Sven also suggest that Daniel uses the definition to determine if the curve represents a function: *There is one and only one function value corresponding to each value of the variable*. Daniel may have used the definition of the concept, but it is unusual for students to do so, according to Sven.

Task 5

To a question from the teacher, whether the graph below represents a function, Emilia answers yes. How can Emilia have reasoned?



A: Emilia may have reasoned that all curves are graphs of a function; therefore, this curve represents a function. (8)

> Curves always express functions. Patrick.

Eight respondents suggest that Emilia reasons that all curves are graphs of a function.

B: Emilia reasons that *x is a function of y*. (3)

> The value of the function is on what is usually called the x-axis and the variable is on what is usually called the y-axis. John

Three respondents suggest that Emilia correctly reasons that *x is a function of y*.

C: Emilia assumes that to one value of x there may correspond several values of y. (1)

> Probably she does not have knowledge of the definition of function and thinks that it is perfectly ok that for a given value of x there are several values of y. Dan

One respondent suggests that Emilia does not have knowledge about the definition of the concept, because she supposes that for a given value of *x* it can correspond several values of *y*.

All interviewed student teachers suggest that Emilia reasons that *all curves you can draw without lifting the pen are graphs of functions*. This erroneous reasoning may be due to the fact that all curves Emilia have met in school have been graphs of functions.

According to Marton's theory of variation, one must see a colour other than green to understand what green is[2]. Therefore, the student must see something that is not a function in order to understand what a function is, Patrick concludes.

Dan suggests that Emilia can reason that *if you rotate the S-shaped curve a quarter of a turn, it looks like an ordinary third-degree curve; therefore, it is a function*. Dan compares this to the fact that when you rotate a triangle you get a congruent triangle.

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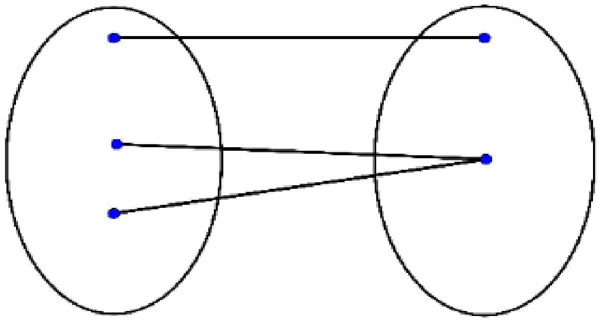
[2] The respondents have had a literature seminar on Ference Marton's book "Necessary conditions of Learning" before the interviews were conducted.

Task 6

To a question from the teacher, whether the diagram below represents a function, Faiza answers no. How can she have reasoned?



A: Faiza does not recognize this representation of a function. (5)

> This is an unusual representation of a function in schools. Students do not meet this visualization very often in secondary schools. I do not think Faiza understands it. It does not look like a graph. Viktor

B: Faiza can read the diagram from right to left because the lines in the diagram lack direction. (5)

> She may have read from right to left and assumed that the domain is on the right, which does not have to be wrong since the teacher has been vague. Eric

All respondents suggest that Faiza may find it difficult to interpret the diagram, either by reading it from right to left, or that she does not recognize this representation at all.

C: Faiza assumes that a function must be injective and hence the diagram does not represent a function. (2)

> She thought that functions must be injective to be called functions. Dan

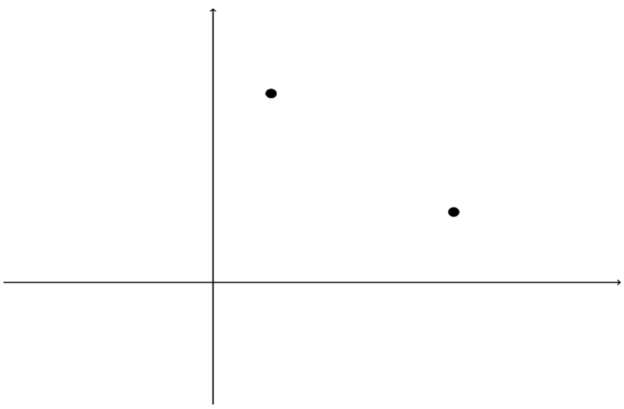
Two respondents suggest that Faiza equates *functions* and *injective functions*; therefore, the diagram does not represent a function.

During the interview, Patrick suggests that Faiza does not know that a function has a domain and a codomain; therefore, she cannot interpret the two ovals in the diagram. Dan, John, and Sven suggest that Faiza has read the diagram from right to left; thus, she has changed places of domain and codomain. Dan suggests that this can be explained by that Faiza has an Arabic origin.

Dan and Patrick suggest that Faiza may reason that the diagram does not represent a function because two elements in the domain are connected to one and the same element in the codomain. During the interview, Dan expresses Faiza's possible reasoning as *maybe she thought that there must not be two of something; maybe it is not allowed to be two different x.*

Task 8

Helena is a teacher of mathematics in upper secondary school. She teaches the course mathematics 1c. During a review on functions, she asks the students how many graphs of a function one can draw through the two given points in the coordinate system below. What possible mistakes do you think the students will make? Why do you think the students will make these mistakes?



A: The students will only draw a straight line through (or between) the two points. (9)

> They will only draw straight lines. They see two points and they are used to connecting them. John

Nine respondents suggest that the students will only draw a straight line through (or between) the two given points.

B: The students will also draw the graph of another function, other than a straight line. (2)

> They will say that there are two functions: linear and quadratic, or a few more.
> Dan

Two respondents suggest that the students will also draw a graph of another elementary functions, such as a quadratic function.

C: The students will draw curves that do not represent functions. (2)

> They will draw a curve with several values of y corresponding to one value of x.
> Fredrik

Two respondents suggest that students will draw curves that do not represent functions, such as an S-shaped curve, similar to the curve in Task 5.

During the interviews, John, Patrick, and Sven suggest that students will only draw a straight line through the two given points because linear functions are the only examples of functions they have met in school.

Dan suggests that students will draw graphs of two classes of functions, linear and quadratic, because the students' image of the concept only consists of the examples they have met in school. Instead of seeing the concept of function as a general concept, the student reasons that the only functions that exist are some classes of elementary functions, according to Dan.

Dan describes a lesson where the purpose was to problematize the concept of function: I drew a circle in a coordinate system on the whiteboard and asked my students: "Does the circle represent a function?" A student responded that it could not be a function because there are no points on it. I thought that I must examine how the student reasons here. Therefore, I constructed the graph of a linear function using three points which I marked strongly and then I drew a straight line through the three given points. The student thought it was a function because there were points on it. Then I formulated the following hypothesis for myself. The student perceives the three points as the function and the line as a filling in between the points. I explained that the line consists of infinitely many points, including the three marked points. My conclusion is that you should erase the marked points, after you have constructed the graph.

5 Discussion

The present study investigates secondary mathematics student teachers' level of KCS regarding the concept of function; in particular, the respondents' suggestions about secondary students' potential difficulties recognizing constant functions and piecewise-defined functions are investigated. Also, suggestions about students' difficulties regarding the one-valuedness property of a function, difficulties related to the various representations, and students use of prototype examples are considered. The results of this study are now discussed in relation to previous research.

5.1 Constant function with an implicit domain

Task 1 in the questionnaire concerns students' potential difficulties in interpreting the algebraic expression $y = 4$. Eight of the ten respondents suggest that students expect an independent variable in an algebraic expression representing a function; therefore, students conclude that the expression $y = 4$ does not represent a function. The respondents' suggestions are consistent with the findings of Tall and Bakar (1992), who report that some students do not recognize constant functions with an implicit domain because they expect an independent variable in the algebraic representation of a function.

The suggestions from the respondents in the present study are also consistent with the findings of Hatisaru and Erbas (2017), who propose that teachers demonstrate an aspect of KCS when they suggest that students may have difficulty in recognizing constant functions with an implicit domain.

The algebraic expression $y = 4$ can be interpreted as a function with an implicit domain, for example, the real numbers. Making this interpretation is not at all obvious; instead, you must learn how to do it. It is an aspect of KCS to recognize the difficulty with an implicit domain of a function; however, no respondent in this study explicitly suggests this in connection with Task 1.

Five respondents in this study suggest that students may perceive the expression $y = 4$ as an equation with one unknown instead of a function. Dan, Patrick, and Sven also suggest that one should distinguish between a variable in connection with functions and an unknown in connection with an equation. This suggestion is consistent with the conclusions of Kilhamn (2014).

5.2 Piecewise-defined function

Only four respondents suggest that students may have difficulties in interpreting the algebraic representation of the piecewise-defined function in Task 2. The domain of the piecewise-defined function is split into two parts. However, no respondent in this study explicitly mentions the difficulty with a split domain, although Patrick suggests that students may reason that *a function must be represented with one expression only*. His suggestion about students' erroneous reasoning about piecewise-defined functions is consistent with the findings of Vinner and Dreyfus (1989), who report that some college students propose that a function cannot be defined by different expressions on different subdomains. This result is also consistent with the findings of Hatisaru and Erbas (2017). Therefore, we conclude that it is an aspect of KCS to know about the difficulty with a split domain of a function.

Some of the college students in Vinner and Dreyfus (1989) propose that a function must be continuous. A teacher in the study of Hatisaru and Erbas (2017) suggests that students think that the graph of a function must not be disconnected. These findings are consistent with the suggestions of eight respondents in this study who suggest that students only know about continuous functions. Nevertheless, these respondents take for granted that students can interpret the algebraic representation of the piecewise-defined function, translate it to a graph and discern a jump in the graph. However, if students can interpret the algebraic representation of a piecewise-defined function, then they have probably studied such functions before; therefore, they presumably know about discontinuous functions too. One difficulty in interpreting the algebraic representation of a piecewise-defined function is what a rule of a function is. Knowledge of this difficulty is an aspect of KCS; however, no respondent in this study suggests this.

5.3 The graph of a function

Seven respondents in this study suggest that the student reasons that the curve in Task 4 is one-valued; therefore, the student concludes that it represents a function. On the other hand, six respondents suggest that the student reasons that the curve in Task 4 looks like the graph of a function, for example, a third-degree curve; hence, the student concludes that it represents a function without checking the one-valuedness property. Eight respondents in this study suggest that students suppose that every curve is the graph of a function. Hence, students draw the erroneous conclusion that the S-shaped curve in Task 5 is the graph of a function.

Six respondents in this study suggest that the curves in Task 4 and Task 5 look like the graph of a function. Their suggestions are consistent with the findings of Tall and Bakar (1992), who report that students – when they try to decide whether a given relation represents a function – rely on familiar examples, such as polynomials. This familiarity evokes the concept of function (ibid.).

5.4 Arrow diagram

The diagram in Task 6 can be interpreted as representing a function or as a relation that is not a function, depending on whether you read it from left to right or vice versa. It is drawn without arrows in order to open up for the possibility to read it from right to left, as five respondents in this study suggest that students may possibly do.

All three parts of a function — rule, domain and codomain — are visible in an arrow diagram. All the respondents in this study demonstrate an aspect of KCS when they propose that students may have difficulties interpreting the diagram in Task 6, for example interpreting the two ovals. Patrick also suggests that some students do not even know that a function has a domain and a codomain; for them it is impossible to interpret the diagram. Also, Dan, John, and Sven discuss the possibility that Faiza has changed places of domain and codomain and read the diagram from right to left. Patrick, Dan, John, and Sven demonstrate an aspect of KCS when discussing the difficulty of identifying the domain of a function.

If students can interpret arrow diagrams, they are useful for illustrating the one-valuedness property of a function. With this use of arrow diagrams in mind we can interpret what Dan means when he writes *Faiza thought that functions must be injective* in his questionnaire. We interpret his statement as communicating an idea to the interviewer between two mathematicians. Dan does *not* take for granted that Faiza has acquired the concept of injective function; instead he is describing a source of Faiza's misconception when he says that Faiza reasons that *it is not allowed to be two different x*. In this quotation Dan is referring to the two points in the left oval that are connected to one and the same point in the right oval. In these two quotations he describes the confusion of the idea of injectivity with the one-valuedness property of a function as a possible source of the difficulty in interpreting an arrow diagram. With this interpretation Dan demonstrates an aspect of KCS.

5.5 Prototype examples

Some students use *prototype examples* of functions instead of the definition when they try to decide if a given example represents a function (Schwarz & Hershkowitz, 1999). The purpose of Task 8 in the questionnaire is to investigate whether student teachers know this. Nine respondents in this study suggest that the students in Task 8 will only draw a straight line through the two given points in the coordinate system. These respondents demonstrate an aspect of KCS, because their proposals are consistent with the conclusion of Markovits et al. (1986) who report that secondary school students drew mostly linear functions in connection with a similar task.

6 Conclusions and implications

Teachers' ability to anticipate students' misconceptions regarding the concept of function, is an aspect of KCS, which in turn is part of MKT (Nyikahadzoyi, 2015). Therefore, teachers need KCS to help their students resolve misconceptions about the concept of function.

Compared to the findings of previous research on the sources of students' difficulties with the concept of function, some of the respondents sometimes provide reasonable suggestions about the sources of students' difficulties regarding the concept of function. For example, all the four interviewed respondents demonstrate an aspect of KCS when they suggest that students may have difficulties identifying the domain of a function in connection with interpreting an arrow diagram as a function. Or, for example, when eight respondents suggest that students only know about continuous functions.

However, some of the respondents' level of KCS regarding the concept of function is not sufficiently developed, for example: Two of the respondents never suggest that students can expect an independent variable in an algebraic expression representing a function. No respondent suggests that one source of students' difficulties with a constant function with an implicit domain is the missing domain. As many as six respondents take for granted that students can interpret the algebraic representation of a piecewise-defined function and translate it into a graph. Only six respondents demonstrate an aspect of KCS when they suggest that students do not always check the one-valuedness property of a function when determining whether a given curve represents a function; instead students use familiar examples of functions.

Teachers need to understand students' ways of thinking in order to help and guide them in their knowledge construction (Even & Tirosh, 1995). Because some of the respondents' level of KCS regarding the concept of function is not sufficiently developed, they may face difficulties in helping and guiding students in their future teaching. Therefore, teacher education needs to facilitate the development of student teachers' level of KCS. This can be achieved by including dialogue seminars in teacher education where the sources of students' misconceptions can be discussed. This may allow student teachers to develop the knowledge required to help students to overcome their misconceptions, for example, that the algebraic representation of a function must include an independent variable, or that a function must be represented with one expression only.

Researchers assume that the difficulties with the rule and the domain of a function manifest themselves in connection with, for example, piecewise-defined functions and constant functions with an implicit domain (e.g. Hatisaru & Erbas, 2017; Vinner & Dreyfus, 1989). Only four respondents in this study suggest that students can have difficulties interpreting the algebraic representation of a piecewise-defined function, because they assume that a function must be represented with one and only one expression. Students may interpret this representation as several rules, and not *one* rule (Vinner & Dreyfus, 1989). Therefore, it is important – in teaching and in teacher education – to emphasize that a rule cannot always be defined by one algebraic expression only. Also, Even (1993) emphasizes the arbitrariness of a rule of a function.

Furthermore, when representing functions algebraically, teachers can make an implicit domain visible, just by explicitly defining a domain. This is especially important for constant functions with an implicit domain, because the independent variable is not present in the algebraic expression. However, no respondent in this study explicitly suggests the difficulty with an implicit domain in connection with the algebraic representation of a constant function with an implicit domain.

It is reasonable to assume that in-service teachers continue to develop their level of KCS while teaching mathematics to students, therefore, we propose further research on in-service teachers' level of KCS regarding the concept of function. Because teachers' KCS and KCT are interrelated, we also propose further research on in-service teachers' *knowledge of content and teaching* (KCT) regarding the concept of function; in particular, teachers' choices of appropriate representations and knowledge of advantages and disadvantages of the various representations as well as how to sequence the content in the teaching can be investigated.

Since the construct MKT is mainly derived from elementary school level, concerns about how transferable it is to secondary school level have been raised (Speer, King, & Howell, 2014). Therefore, we propose these concerns regarding transferability as a topic for further research.

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Appendix A

A summary of the respondents' suggestions on the tasks in the questionnaire.

	1	2	4	5	6	8
Bo	B	B	B	A	A	A
Dan	AB	B	B	BC	BC	AB
Eric	AB	A	B	A	B	AC
Fredrik	B	B	A	A	B	AC
John	A	B	AB	B	B	A
Patrick	A	A	AB	A	A	A
Rickard	A	B	A	AB	A	-
Sven	AB	AB	AB	A	BC	A
Tom	A	B	B	A	A	A
Viktor	A	AB	A	A	A	AB