Identifying and promoting young students' early algebraic thinking

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Algebraic thinking is an important part of mathematical thinking, and researchers agree that it is beneficial to develop algebraic thinking from an early age. However, there are few examples of what can be taken as indicators of young students' algebraic thinking. The results contribute to filling that gap by analyzing and exemplifying young students' early algebraic thinking when reasoning about structural aspects of algebraic expressions during a collective and tool-mediated teaching situation. The article is based on data from a research project exploring how teaching aiming to promote young students' algebraic thinking can be designed. Along with teachers in grades 2, 3, and 4, the researchers planned and conducted research lessons in mathematics with a focus on argumentation and reasoning about algebraic expressions. The design of teaching situations and problems was inspired by Davydov's learning activity, and Toulmin's argumentation model was used when analyzing the students' algebraic thinking. Three indicators of early algebraic thinking were identified, all non-numerical. What can be taken as indicators of early algebraic thinking appear in very short, communicative micro-moments during the lessons. The results further show that the use of learning models as mediating tools and collective reflections on a collective workspace support young students' early algebraic thinking when reasoning about algebraic expressions.

Keywords: early algebraic thinking, learning activity, mathematical thinking, primary school, Toulmin's argumentation model

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1 Introduction

This article contributes an analysis of young students' potential to develop algebraic thinking. In a research review on mathematical thinking, Goos and Kaya (2020) point out that two broad aspects of mathematical thinking are mathematical problem-solving and mathematical reasoning. Translated to algebra, problem-solving and reasoning are activities in which indicators of algebraic thinking can be explored in the form of students' communicative actions. In this article, I will consider algebraic thinking as a part of mathematical thinking (e.g., Blanton et al., 2015; Cai & Knuth, 2011; Kaput, 2008; Kieran, 2004).

A basic condition for developing knowledge in different areas of mathematics is students' theoretical understanding of algebra (Davydov, 1990, 2008). Thus, algebra has a special position in mathematics since it is found in all other mathematical areas.

Both general reasoning in arithmetic, proof in number theory, and geometric formulas for area and volume use algebra as a working tool. In the form of equations, algebra is used in problem-solving in almost all mathematics. It is argued that a robust knowledge of algebra makes it easier for students to succeed in their further studies (Kieran et al., 2016; Matthews & Fuchs, 2020; see also Schoenfeld, 1995). However, many students find algebra difficult to learn (e.g., Carraher & Schliemann, 2007; Kieran, 2007; Matthews & Fuchs, 2020), and many teachers find it difficult to teach (Chick, 2009; Kilhamn et al., 2019; Röj-Lindberg, 2017; Röj-Lindberg et al., 2017). Algebra is also the mathematical area in which students perform poorly in all Nordic countries (Hemmi et al., 2021). Since it is regarded as challenging mathematical content, in the Western world algebra has tended to be introduced rather late, often as late as lower secondary school (see e.g., Bråting et al., 2019; Hemmi et al., 2021; Kilhamn & Röj-Lindberg, 2019; Stacey & Chick, 2004).

However, for several decades now there has been substantial interest in the youngest students' algebraic thinking, including their reasoning and problem-solving capabilities (e.g., Blanton et al., 2015; Bråting et al., 2019; Eriksson et al., 2019; Kaput, 2008; Lins & Kaput, 2004; Schmittau, 2004, 2005). Kaput et al. (2008) argue that it is not only possible but also beneficial to early develop students' algebraic thinking, in addition to their arithmetic thinking. In today's Nordic school mathematics curricula, Grades 1–6, algebraic content such as patterns, equalities, and equations are introduced (Børne- og Undervisningsministeriet, 2020; Skolverket, 2019; Utbildningsstyrelsen, 2020; Utdanningsdirektoratet, 2020; see also Bråting et al., 2019).

In the field of early algebra, reference is made to the so-called Davydov curriculum[1] in mathematics as a promising model for the development of algebraic thinking (Kaput et al., 2008; see also Cai & Knuth, 2011; Schmittau, 2004, 2005; Venenciano & Dougherty, 2014). This curriculum, with its roots in the Vygotsky tradition, is based on learning activity aimed at developing students' theoretical thinking in mathematics, and foremost their algebraic thinking (Davydov, 1990, 2008; Schmittau, 2004, 2005). Central to the Davydov curriculum is mediating tools, what he calls learning models, and collective reflections, which are used as a means for supporting students' theoretical work (Davydov, 2008; Gorbov & Chudinova, 2000; Repkin, 2003; Zuckerman, 2003).

[1] *The Davydov curriculum* is also referred to as *Davydov's programme,* and *the El'konin-Davydov curriculum (ED curriculum).*

Less is known about how young students' algebraic thinking emerges during classroom work and how it can be identified (Goos & Kaya, 2020). There is thus a need for empirical examples of how early algebraic thinking among young students can be identified and promoted, and what in the design of tasks, tools, and communicative resources has the potential to enhance students' early algebraic thinking (e.g., Goos & Kaya, 2020). Given that thinking cannot be analyzed as such (Radford, 2008a, 2010), there is a need to use, for example, the students' tool-mediated communicative actions as indications of their algebraic thinking.

1.1 Aim and research questions

In this article I analyze young students' communicative actions on algebraic expressions[2] to identify indicators of early algebraic thinking and discuss what in a learning activity promotes students' opportunities to explore algebraic expressions. The aim is specified in two research questions (RQs):

RQ1 What in young students' tool-mediated communicative reasoning can be taken as indicators of their early algebraic thinking?

RQ2 What in the learning activity promotes young students' early algebraic thinking when exploring algebraic expressions?

2 Background and research on algebraic thinking

Several researchers in the field of early algebra argue that challenges concerning algebraic thinking may be due to algebra usually being introduced through arithmetic, for example in the form of tasks focusing on equalities in which the value of an unknown number is requested (e.g., Kieran, 2006; Kieran et al., 2016; see also Lins & Kaput, 2004; Stacey & MacGregor, 1999). This may be a reason that difficulties arise later in algebra learning, where students often get stuck on numerical solutions (see e.g., Kaput, 2008; Kieran, 2006; Radford, 2010). Therefore, it is argued that the introduction of algebra should promote algebraic thinking from the beginning of primary school (Lins & Kaput 2004; Roth & Radford, 2011). Kieran (2004) suggests that students need to work theoretically in different ways at an early stage. Thus, they

[2] In this article, an algebraic expression refers to a meaningful composition of mathematical symbols (Kiselman & Mouwitz, 2008). This implies, for example, that $x + y - z$ and $yx + z$, but also the inequality $x < y$ and the equality (or equation) $x = y + z$, are expressions (James & James, 1976). In the study on which this article is based, we have used algebraic expressions in the form of equalities of the type $a = b + c$.

need to encounter tasks and problems that promote algebraic thinking; for example, they need to have the opportunity to explore algebraic structures (Blanton et al., 2015; Bråting et al., 2018; Kieran et al., 2016).

To promote the development of algebraic thinking, teachers need to create conditions for students to develop abilities such as reasoning algebraically, making algebraic generalizations, and using algebraic representations, rather than teaching several procedures (Greer, 2008; Kaput, 1999; Usiskin, 1988). Mediating tools can furthermore play a crucial role in developing a so-called "non-counting" approach, whereby students work with problem-solving tasks (Schmittau, 2004; Venenciano & Dougherty, 2014). Algebraic thinking can involve theoretical work with letter symbols or other relational resources in the students' analysis of, for example, relations between quantities, structures, and patterns. This also includes working with justifying and proving (Kieran, 2004).

Kaput (2008, p. 11) highlights two core aspects that account for algebraic thinking: "(A) Algebra as systematically symbolizing generalizations of regularities and constraints. (B) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems". Thus, the development of algebraic thinking includes, among other things, the exploration of general, fundamental, and theoretical relationships and structures (see also Blanton et al., 2015; Davydov, 1990, 2008; Venenciano & Dougherty, 2014).

Matthews and Fuchs (2020) point out the relational aspect in the equal sign as an especially important component of algebraic thinking, referring to it as a "big idea" in mathematics (Matthews & Fuchs, 2020, p. e15). Instead of understanding the general structure, students often understand it as an operator that implies a sum or a result. Thus, the students need to develop a relational view of the equal sign and interpret it as "the same as" (Matthews & Fuchs, 2020, p. e15; see also MacGregor & Stacey, 1997; Warren & Cooper, 2009). Schmittau and Morris (2004) state that it is possible for young students, by comparing quantities, to theoretically work with inequalities and equalities. When children write

> "If $C<P$ by B, then $C = P-B$ and $C+B=P$"; the notation indicates that they can move from an inequality to an equality relationship by adding or subtracting the difference, and that addition and subtraction are related actions. (Schmittau & Morris, 2004, p. 81)

Also, Blanton et al. (2015) point out the relational understanding of the equal sign as important and include this in the big idea of *equivalence, expressions, equations, and* *inequalities* (EEEI). Further, Blanton et al. (2015, p. 43), as part of EEEI, include "representing and reasoning with expressions and equations in their symbolic form and describing relationships between and among generalized quantities that may or may not be equivalent".

Ventura et al. (2021) argue that young students can use the function of variables in algebraic expressions (see also Venenciano et al., 2020). In a Nordic context, however, few studies have explored algebraic thinking with symbols and relational material. Eriksson et al. (2019) used non-numerical examples, in the form of $a = b + c$, when introducing algebraic expressions in grades 1 and 5 (see also Eriksson & Jansson, 2017; Wettergren et al., 2021). Given that algebraic expressions are central aspects of algebra, the use and understanding of indeterminate quantities is considered crucial for the development of algebraic thinking (Ventura et al., 2021). In the study on which this article is based, the tasks have been constructed in line with Kaput's (2008, p. 13) description: "the initial symbolization uses letters to denote quantities, thereby embodying generality in the symbolic expression of specific (but unmeasured) cases involving, say, comparisons of lengths".

3 Learning activity as a theoretical framework

According to Vygotsky (1986), a prerequisite for developing theoretical thinking is a teaching in which children are allowed to encounter scientific (theoretical) concepts at an early age, compared to being introduced to everyday (empirical) concepts. With reference to Vygotsky, Schmittau (2004, p. 39) argues that "in order to learn mathematics as a *conceptual system*, it is necessary to develop the ability to think *theoretically*". Thus, students need to develop theoretical thinking early, through a teaching that offers them opportunities to engage in work with concepts and their relations and structures.

The Davydov curriculum in mathematics and learning activity is based on the idea of "ascending from the abstract to the concrete" (Davydov, 2008, p. 106). He argues that students first need to work with general structures and relations in, for example, algebraic expressions in order to later use them in concrete numerical operations. A basic principle of learning activity is that theoretical thinking related to mathematical concepts needs to be explored with the help of mediating tools, in learning activity referred to as *learning models* (Davydov, 2008; Gorbov & Chudinova, 2000; Repkin, 2003). Learning models can be seen as materialized representations of the abstract (Repkin, 2003). These can be constructed of physical representations (e.g., Cuisenaire rods[3]), symbols (e.g., in the form of variables), schemes represented with lengths (I——I), and graphs (e.g., in coordinate systems). According to Davydov, learning models have different functions that aim to enable students to theoretically explore the abstract (structural) aspects of a given object of knowledge (Davydov, 1990, 2008; Davydov & Rubstov, 2018; Gorbov & Chudinova, 2000). Davydov emphasizes that "not just any representation can be called a learning model, but only one that specifically fixates the universal relation of some holistic object, enabling its further analysis" (Davydov, 2008, p. 126). The intention of a learning model is to make certain aspects of an object visible, and it functions as a tool when students work on a problem. A learning model can also function as a tool for classroom communication (Davydov, 2008). Radford (2008b, p. 219) argues that the tools "are not merely aids: their mediating role is such that they *orient* and *materialize* thinking and, in so doing, become an *integral* part of it". In other words, learning models can visualize students' thinking and thus constitute a mediating tool in the work with concepts.

Another basic principle in learning activity is that students be given the opportunity to participate and engage in *collective reflections* (Zuckerman, 2003, 2004). Thus, the mathematical content can be made visible, explored, and developed as a conceptual understanding. Students' experienced motive for engaging in theoretical work can be made possible when groups of individuals are allowed to work together and share or borrow each other's experiences and knowledge (Vygotsky, 1986; Zuckerman, 2004). Thus, reflection is not seen as an individual process but rather takes place collectively. The starting point for collective reflections is that students, by engaging with other students' suggestions and explanations, can understand their own thinking (Zuckerman, 2003).

To realize a learning activity, for example, regarding algebraic expressions, the teacher has to enable and pursue an elaborative discussion among the students (Zuckerman, 2004). When planning for a learning activity, the teacher's choice of a task framed as a problem situation, as well as considerations of how the structural aspects can be visualized, are crucial (Repkin, 2003). Thus, students should encounter a problem situation that requires work and that can result in the development of their theoretical thinking. Such a problem situation must be perceived by the students as meaningful; that is, they should experience a need to explore the problem. The students' exploration of the problem situation comes about through collective reflections together with the teacher and can take the form of class discussions. These

[3] Cuisenaire rods are a relational laboratory material that consists of rods of different lengths and colors, with each length being a certain color (Küchemann, 2019).

can take place on what can be understood as a collective workspace, for example when the representations and symbols are displayed on a whiteboard, which has a decisive function in this respect (Eriksson et al., 2019).

4 Method

This article is based on a study which took the form of a design experiment (e.g., diSessa & Cobb, 2004). In this study, we, four researchers and five teachers worked collaboratively. That is, the researchers and teachers iteratively planned, adjusted, and refined the lessons and problem situations together (Carlgren, 2012; Eriksson, 2018). The study's focus was on developing mathematics teaching; more precisely, on promoting young students' early algebraic thinking and their reasoning about algebraic expressions through a collective and tool-mediated teaching situation.

4.1 Data

The study was conducted at two municipal schools, with 550 and 1150 students respectively, and from preschool class to Grade 9. The five participating teachers had between 15 and 23 years of teaching experience. The all signed up to the project voluntarily. Four of the teachers had a Grade 1–6 teaching qualification and one had a Grade 4–9 teaching qualification. Forty-two students across grades 2 to 4 participated in the various research lessons (Table 1).[4]

Table 1. Research lessons conducted 2015–2017

School year	Grade	Students in research lesson 1	Students in research lesson 2	Students in research lesson 3	Total students in the research lessons
2015–2016	2[5]	6[6]	6		12
2016–2017	3	6	6	6	18
2016–2017	4	4	4	4	12

Inspired by learning study (Marton, 2005, 2015; Runesson, 2017), eight research lessons were conducted in grades 2, 3, and 4 (students aged 7–10) at two schools during the period 2015–2017. One research lesson cycle in each grade was held with

[4] The research lessons were conducted within the context of the mathematics network at *Stockholm Teaching & Learning Studies* (STLS).

[5] Only two research lessons were carried out in Grade 2

[6] Research lesson 1 in Grade 2 has been excluded due to administrative complications.

different groups of students in the same grade. The cycles were conducted iteratively over the school year. The research lessons, lasting between 26 and 44 minutes, were video recorded. Altogether, the lessons amounted to 251 minutes. None of the student groups had previously worked with algebraic expressions or formal equations during the school year in which the data were collected. The data in this article are taken from the second research lesson in each grade since the students' collective reasoning on the general structures and relational aspects of the algebraic expressions was rich.

The jointly planned lessons, all in the form of whole-class discussions, were conducted by one teacher in each grade. The other teachers in attendance were responsible for the data collection, such as video recording and observation. As the focus of the recording was the joint activities on the whiteboard in front of the class, the video camera was mostly pointed at the whiteboard. The research lessons were transcribed in their entirety according to Linell's (1994) description: word-for-word, speech-neutral text, organized in dialogic form.[7] Interaction in the form of gestures and concrete manipulations, when these appeared in the video, are also described in brackets in the excerpts since they can be seen as part of the argumentation (Nordin & Boistrup, 2018). The students were given fictitious names.

4.2 The design of the research lessons

The design of the research lessons was inspired by learning activity (Davydov, 2008) and the previously mentioned concepts of learning models (Repkin, 2003) and collective reflections (Zuckerman, 2003). In addition, special attention was paid to enabling joint work on the collective workspace (Eriksson et al., 2019; cf., Liljedahl, 2016); that is, on the classroom whiteboard.

The overarching aim of each lesson was for the students to discern the relations between quantities, structures, and general patterns in algebraic expressions. Therefore, the problem situations they were to explore and reason in relation to consisted of contrasting examples of visualizations of algebraic expressions. Having the students encounter problem situations with alternative solutions and asking them to reflect on and explain someone else's solution made it possible for them to take another person's perspective (Zuckerman, 2004). Also, in the design of the teaching, learning models were used as mediating tools. These took the form of Cuisenaire rods

[7] As the research lessons were conducted in Swedish, the transcripts have been translated into English.

(Küchemann, 2019).[8] Line segments drawn on the whiteboard and symbols for variables were also used as learning models. To promote students' algebraic thinking, the teachers planned the possible types of responses they could employ depending on the situation and took into account that students' discussions could either stop or take a less desirable direction than had been hoped. Considering that students' participation in a learning activity should be characterized by their agency, how the teacher responds is important. For example, asking "how did xx think here?" instead of addressing the individual student and asking, "how did you think here?" can lead to different results.

The overall structure of the research lessons was the same. The lessons started with the presentation of a learning model in the form of line segments or Cuisenaire rods visualizing a relation that was to be collectively reflected on. The teacher worked with the students at the whiteboard. Also, each student had access to tools in the form of rods on the table in front of them. The teacher was responsible for maintaining the students' collective reasoning through questions and provocations. Occasionally, the students approached the whiteboard when they were to present their suggestion or solution to the given problem situation.

4.3 Analysis

Toulmin's model of argumentation (Toulmin, 2003) was used to analyze the class discussions for possible indications of algebraic thinking. Toulmin's model is a theoretical model of an argument and has most commonly been used in research on interaction within mathematics education, for example, proof (e.g., Hemmi et al., 2013), often in the reduced version introduced by Krummheuer (1995). The reduced model consists of four elements, three of which—claim, data, and warrant—are regarded as the core of an argument, along with a potentially fourth element, backing. A claim is a statement that is grounded in data, and the warrant functions as a bridge between data and claim. According to Toulmin (2003), the data supporting the claim can answer the question "What have you got to go on?" (Toulmin, 2003, p. 25) and the warrant would answer "How do you know?" (Toulmin, 2003, p. 210). In an analysis of the interaction, the argument and each of the four elements can be created by more than one individual. The elements do not need to be expressed in a specific order and can be expressed in many ways, for example verbally, with written symbols,

[8] The Cuisenaire rods used in the study were comprised of various materials, some wooden and some magnetic, the latter designed to be used on a whiteboard.

drawings, or gestures, or using manipulatives, in this case Cuisenaire rods (Nordin & Boistrup, 2018).

The analysis began with a reading of the transcripts for the research lessons in each of grades 2, 3, and 4 in their entirety. In each transcript, sequences were identified. Here, a sequence is the time between the teacher's presentation of a new problem situation with an algebraic expression or a learning model visualizing an algebraic expression, and the presentation of the next problem situation. Within each sequence, I searched for various reasonings that used arguments with claims and highlighted them. I then searched for data supporting each claim. If I found data supporting the claim, I searched for the warrant. In some cases, the students also gave non-mathematical suggestions; for example, the variables needed to be in alphabetical order, or a specific value had to be requested. Therefore, an additional delimitation of an argument concerning algebraic thinking was that it should focus on the relational and general aspects. The initial analysis was also discussed with the participating teachers, a process described by Wahlström et al. (1997) as *negotiated consensus*. All video recordings and transcripts were reviewed.

When the elements of an argument were identified, a reconstructed argument was written. To clarify the reconstructed argument in the excerpts below, I have written the elements claim, data, and warrant in brackets. I present the excerpt, followed by a reconstructed argument following Toulmin's reduced model. After the reconstructed argument, I interpret the indicator of early algebraic thinking.

Reconstructing the arguments indicated early algebraic thinking in the students' arguments. Following Radford's (2008a) idea that students' communicative actions can be understood as a form of reflections of thinking, aspects of the students' theoretical work were identified. However, while students' communicative actions are not to be equated with theoretical thinking, they can serve as indicators of theoretical thinking, in this case early algebraic thinking.

5 Results

I present the results from the analysis of the research lessons below. Three empirical examples, not age-specific, are chosen to exemplify indications of students' early algebraic thinking, one each from grades 2, 3, and 4. The chosen examples represent the focus in the class discussion in each grade.

5.1 Grade 2: Establishing equalities

In Grade 2, the teacher introduced algebraic expressions by presenting a make-believe café where the students were to work. This café sold goods such as buns, chocolate bars, and sandwiches. Initially, the students were to construct a price list for the goods. There was no ordinary money; the prices of the goods had to be represented by something else and the teacher suggested Cuisenaire rods. The teacher said: "in this café that you work in, a bun costs a purple rod, like this [holding up a purple rod] … You do not have regular money here; we only have rods like this." The rods were on a table in front of the students, and the teacher had placed corresponding rods in the form of magnetic strips on the whiteboard. In establishing the price list, the teacher said that a bun cost one purple rod (Figure 1).

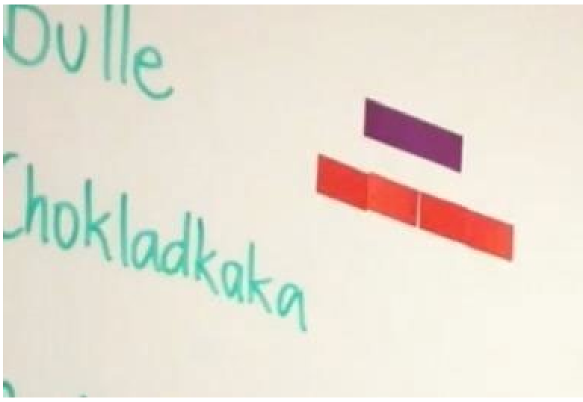

Figure 1. Two buns cost four red rods.[9]

Excerpt 1 (60 seconds)

Teacher:	So, if you were to come to me and buy a bun, what would you pay then? Karin, what would you pay?
Karin:	A purple [rod]. **[data]**
Teacher:	A purple [rod]. And Seydou [the teacher points to the student], what do you think? What would you pay for a bun?
Seydou:	Are there also others [referring to the rods on the whiteboard]?
Teacher:	How do you mean?
Seydou:	Could it also be that money [referring to other rods on the whiteboard] … with some others too?
Teacher:	The bun costs just that [points to the horizontal purple rod]. **[data]** But did you want to pay with something else?
Seydou:	Yes.
Teacher:	Some other rods like this? [points to all the rods on the whiteboard]. Which [rod] would you like to pay with, then?
Seydou:	Four red [rods]. **[claim]**

[9] "Bulle" and "Chokladkaka" are the Swedish words for "bun" and "chocolate bar", respectively.

Teacher:	Four such [places four red rods, horizontally, centered under the purple rod to the right of the word "bun"]. How did you think now, can ... why do you want to pay with four red [rods]?
Seydou:	Because look, two are one ... [in his intonation indicating that two red rods correspond to one purple rod] **[warrant]** ... I wanted to buy two buns [with an indication of emphasis in his tone]. **[claim]**

First, Seydou indicates an understanding that a bun which costs one purple rod corresponds with two red rods. Second, in his argument he elaborates with the new value and currency of the bun (two red rods) when he argues that he intends to buy two buns and therefore wants to pay with four red rods. With his claim and warrant, Seydou demonstrates an understanding of the principle of equality. In the following, the reconstructed argument for why two buns cost four red rods is presented (Figure 2).

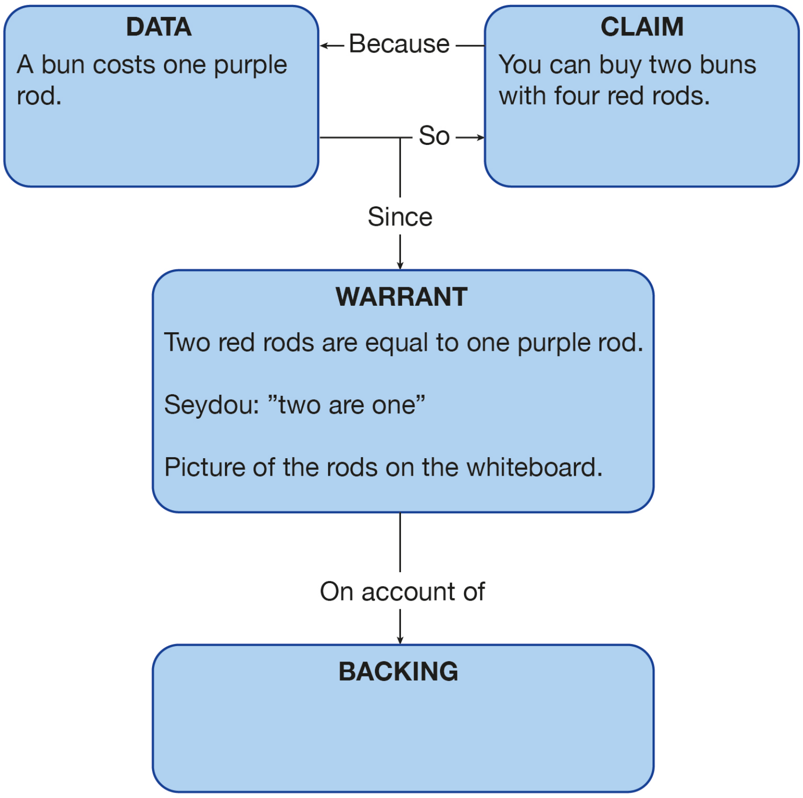

Figure 2. Reconstructed argument for why two buns cost four red rods.

The reconstructed argument shows how Seydou implicitly uses the learning models when pointing out the relation between the different rods, with their lengths set to have a non-numerical value, by choosing rods that equal the purple one. The indicator of early algebraic thinking in this case involves establishing equalities.

5.2 Grade 3: Adjusting inequalities to equalities

In the Grade 3 research lesson, the group had previously worked on a problem situation in which they created equalities that would correspond to the expression $z = x + y$. This was done using learning models where the students worked with rods on the tables and the teacher drew corresponding vertical line segments on the whiteboard. Based on the student examples, several alternatives were created, all of which corresponded to the algebraic expression. In connection with this, the group had a shorter discussion where it was stated, among other things, that the constructions needed to have the same length **[data]**. The teacher then drew new line segments on the whiteboard, with the one on the left representing *z* and on the right a combined line segment representing *x* and *y* (Figure 3).

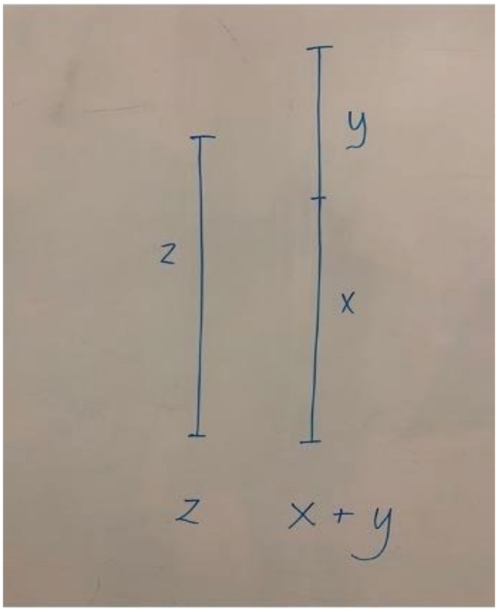

Figure 3. Constructed model of line segment *z* and line segment *x* and *y*.

This constructed a learning model in which the line segments were no longer equal **[data]**, even though they related to the same algebraic expression as before ($z = x + y$). The students collectively expressed that this was not correct and came up with some suggestions for how to make the line segments and the expression match. One of the students, Sisay, was given the floor:

Excerpt 2 (1 minute 16 seconds)

Sisay:	You could also ... above z, draw a small "block" [referring to a small rod shaped like a cube on the table in front of her] that we name w or something, eh ... **[starting a claim]**
Teacher:	[Measures with thumb and index finger as a distance above z so the line segment on the left will be as long as those on the right] so that they become equal [draws a new line segment, which she calls w so that the left now consists of two line segments (z and w)] **[warrant]**
Sisay:	Because then ... in that case it will be z plus w is equal to x plus y. **[claim]**
Teacher:	Like this, z plus w is equal to x plus y [writes $z + w = x + y$ below the line segments on the whiteboard] (see Figure 4). Mm. What does Elsa think of this?
Elsa:	Eh ... yes.
Teacher:	Or do you think, what do you think? Do you agree ... do you understand what Sisay means? Do you think that one can do this?
Elsa:	Mm.
Teacher:	Why can you do that, then? You know, yes ... but this is what we think we can do, but what is it that makes us think we can do it like that? What makes it feel like it's right when you do it like that? Lisa.
Lisa:	That they'll have the same length. x plus ... **[warrant]**
Teacher:	They're the same length [points to w and z in the line segments to the left].
Lisa:	Yes, x plus y and z plus w will be equal in length. **[warrant]**

An indication of early algebraic thinking is when Sisay adjusts the learning models (Figure 4), that is, the line segments on the whiteboard, from an inequality to an equality. In doing so she adds a new symbol, which she decides to name w.

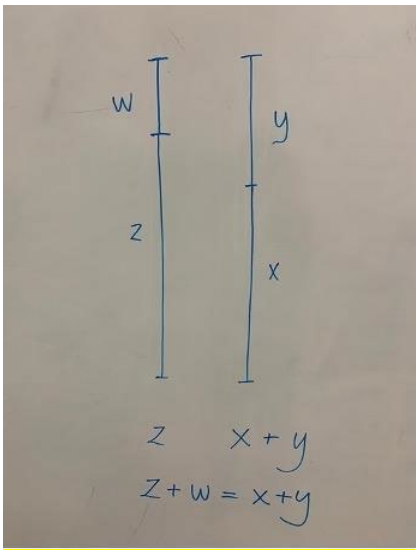

Figure 4. The expression $z + w = x + y$ and adjusted model for the corresponding expression.

Thus, the line segments need to have the same length in order to be equal. When adding a new length and naming it "w" the line segments become equal. However, it is not enough to only adjust the line segments, also the expression needs to be complemented to $z + w = x + y$ to correspond. In the following, the reconstructed argument for why you need to add "w" is presented (Figure 5).

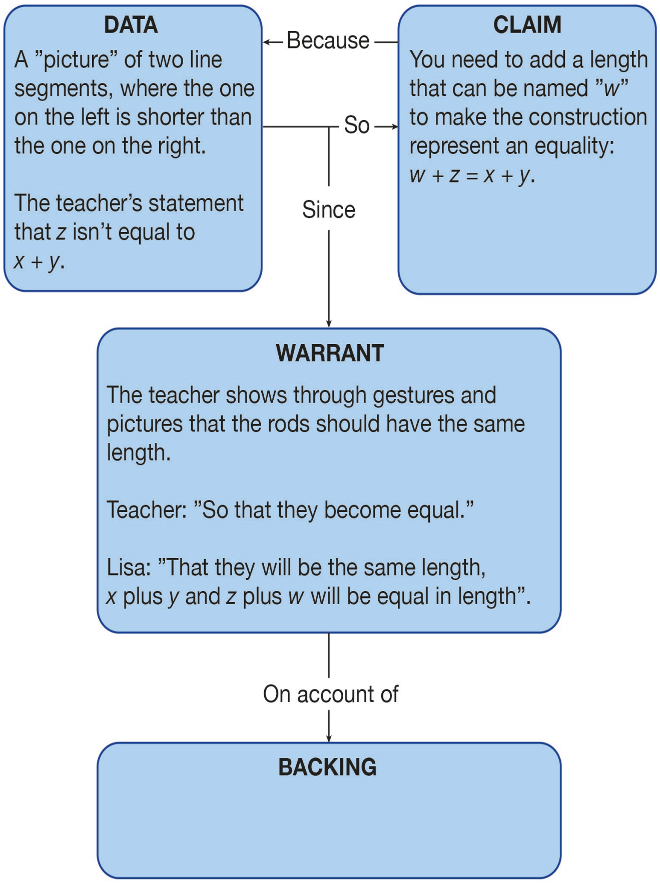

Figure 5. Reconstructed argument for why you need to add "*w*".

The reconstructed argument visualizes Sisay's and Lisa's respective understandings of both the equal sign and the functions of variables when adjusting inequalities to equalities.

5.3 Grade 4: Generalizing equalities

During the lesson in Grade 4, the group worked collectively on the algebraic expression $a = b + c$. The teacher wrote the expression on the whiteboard **[data]**, after which the students constructed variations of the expression with the rods, as a learning model, on a table. The students then were asked to display their constructions on the whiteboard (Figure 6).

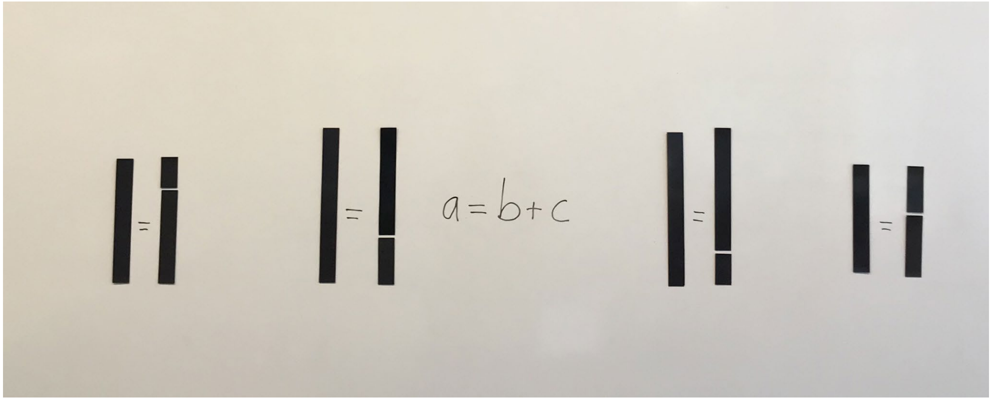

Figure 6. The students' constructions of the expression $a = b + c$.

The first construction to the left and the fourth construction to the right used rods of equal length to represent a, while the second and third constructions in the middle of the whiteboard used rods longer than those in the former constructions (also of equal length) to represent their a. The lengths of the rods representing b and c varied in all constructions. Thus, four different representations of the equality $a = b + c$ appeared **[data]**, all of which were valid representations of the algebraic expression $a = b + c$ **[warrant]**. After a short discussion in which the students expressed that all the alternatives were correct, the teacher asked why a might differ.

Excerpt 3 (50 seconds)

Teacher:	Mm, why might they, the a's, differ [with reference to the fact that there are different large a's in the rod constructions on the whiteboard]? **[data]**
Johan:	Eh ...
Teacher:	Okay, Johan.
Johan:	It depends a bit on what we think a can be ... because we should still build a ... that a ... describe with these, [with reference to the rods on the whiteboard] **[warrant]** that a is equal to b plus c **[claim]**, [referring to earlier discussion] so I suppose we could choose a template ... kind of like this length [refers to a rod he lays in front of him on the table] and then we take b ... in other words, another [puts a rod next to the newly laid rod] ... plus ... we can take something completely different here ... then we choose b plus c [starts adding a third rod above the last laid rod] ... sorry, now I have the wrong rod [referring to the rods he has on the table which result in an inequality], but if you think about mine over there [points to his rod construction on the whiteboard], one ... one b plus c ... one ... so we kind of made a new "a" of two other pieces. **[claim]**

Indication of early algebraic thinking is the reasoning about general structures in Johan's utterances: "It depends a bit on what we think a can be," "I suppose we could choose a template," and "we can take something completely different here." When there are not enough rods on the table, he chooses to relate to his own construction on the whiteboard after initially starting with a randomly selected rod. In the following, the reconstructed argument for why *a* can differ in length and still have the value of *a* is presented (Figure 7).

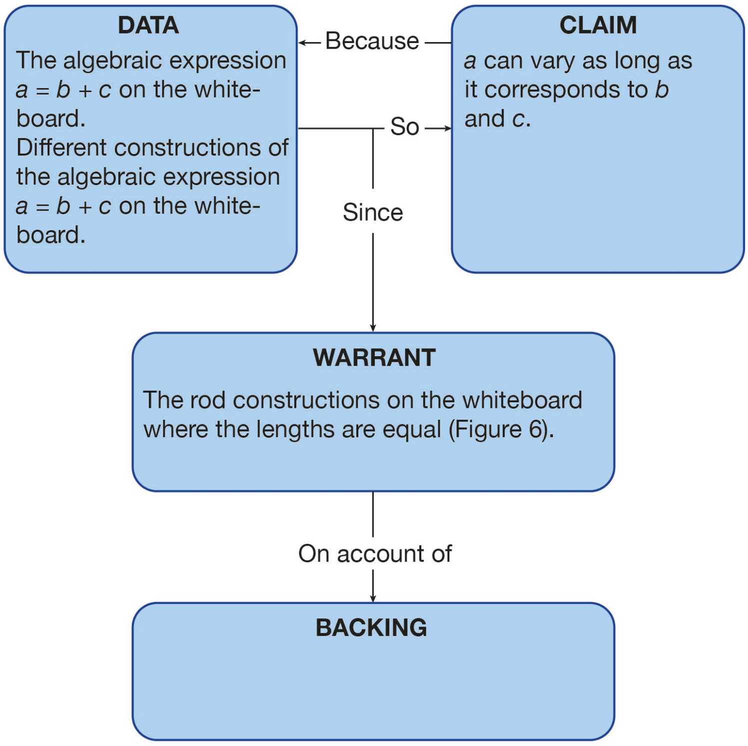

Figure 7. Reconstructed argument for why *a* can differ in length and still have the value of *a*.

The reconstructed argument above shows that the same variable can have different lengths. That is, the student apparently does not need to decide the value of *b* and *c* as he reasons about the algebraic expression without determining the value of the variables. The indicator of early algebraic thinking in this case involves generalizing equalities.

6 Discussion

As presented above, the results give a set of indicators of early algebraic thinking among young students, empirically exemplified in the excerpts. The results also exemplify how early algebraic thinking can be identified. It could be argued that the small number of students in each group in the study does not represent a realistic teaching situation, where the student groups are typically much larger. However, the communicative actions analyzed in this study contribute to knowledge about identified indicators of early algebraic thinking among young students in micro-moments, which can be overlooked in daily teaching in whole-class settings or situations. The results also point to the importance of planning the teaching situation (that is, the problem situation, learning models, and the teacher's responses) to engage the students in an explorative algebraic learning activity.

In the next sections, I discuss the results more closely in relation to the aim of the article and earlier research.

6.1 Indicators of early algebraic thinking

In line with research on early algebra, the results confirm an emergence of young students' early algebraic thinking when working with general structures and relationships in algebraic expressions (Blanton et al., 2015; Bråting et al., 2018; Kaput, 2008; Kieran, 2006; Kieran et al., 2016). Above all, two of the three core concepts of students' early algebraic thinking that Ventura et al. (2021, p. 4) highlight were found: "the relational understanding of the equal sign" and "generalizing and representing indeterminate quantities in algebraic expressions".

However, as the results show, it was only during short moments that an indicator of early algebraic thinking could be identified. The indicators identified in this study are: 1) establishing equalities, 2) adjusting inequalities to equalities, and 3) generalizing equalities. The student in Grade 2, while elaborating on/with the learning models, was able to establish an equality. In doing so, Seydou expressed an understanding of equality when he argued that two buns could be bought with four red Cuisenaire rods. An understanding of the equal sign is something that Matthews and Fuchs (2020) mention as important for students to interpret. Besides exhibiting the important ability to interpret "the same as" (Matthews & Fuchs, 2020, p. e15), the students in Grade 3 also adjusted an inequality given in the learning models by adding a length and naming it with the symbol "w". Also, the expression $z = x + y$ was adjusted to $z + w = x + y$. In doing so, they moved from an inequality to an equality, which according to Schmittau and Morris (2004) is an example of algebraic thinking (see also, Eriksson & Jansson, 2017; Kieran et al., 2016). Johan, in Grade 4, showed indications of Kaput's (2008) core aspects; that is, when generalizing on the equalities he reasoned algebraically. Kieran (2004) highlights justifying and proving as examples of actions that involve algebraic thinking. In addition, the result indicates that the students collectively developed their understanding of the concept of variables through the learning models. While they did not use the term variable, they demonstrated their understanding through everyday language and gestures.

6.2 Promoting early algebraic thinking

The collective reflections in the research lessons were made possible through the joint tool-mediated theoretical work on which the reasoning was based. The learning models visualize, or as Radford (2008b, p. 219) argues "*orient* and *materialize*" the students' theoretical thinking on the relational structure in the algebraic expressions. Further, the learning models on the collective workspace visible to everyone contributed to these reflections (Eriksson et al., 2019). In the collective reflections, the teachers' responses were crucial in establishing and maintaining the learning activity. For example, the teachers were not content when a student gave a correct answer. Instead, they questioned the student's answer by saying "I don't understand," "[c]an this really be true?" or asking another student to explain the given solution. Also, the collective reflections made it possible for the students to, so to speak, borrow knowledge from each other which enabled them to qualify their reasoning.

In the Grade 2 lesson, the teacher used the student's suggestion to challenge him to engage in the theoretical work. That is, the teacher's question "[w]hich [rod] would you like to pay with, then?" required a claim that needed to be substantiated. In Grade 3, the teacher created an example in which the learning model did not correspond to the given algebraic expression. This required that students explore the problem situation, and they collectively manipulated the line segments to create an equality, not by extending the existing line segment named z but by adding a new one that Sisay decided to call w. In Grade 4, the teacher's question along with the students' rod constructions on the whiteboard challenged the students and promoted algebraic thinking. Although all the different examples corresponded to the algebraic expression, the teacher was not "satisfied" with/did not settle for this. Instead, by departing from the students' different constructions, the teacher prompted them to argue for how the different equalities in the rod constructions could all correspond to the same algebraic expression.

6.3 Concluding remarks

Algebraic thinking is a part of mathematical thinking, and the results illustrate how collaborative tool-mediated reflections can promote the development of students' early algebraic thinking. It should be noted that, according to the teachers, the participating students had not worked with this type of algebraic expression previously during the current school year. Moreover, aspects of the teaching situation were new to them. An example of the new teaching situation was that the teacher did not directly confirm whether the students' suggestions were correct. Another example was the problem situations upon which the students were expected to elaborate, since they all consisted of non-numerical but tool-mediated examples. The students were also unaccustomed to collectively working at the whiteboard and elaborating on the theoretical content in front of the student group (Zuckerman, 2004) as well as to working with learning models (Repkin, 2003). The establishment of a learning activity in which students can experience a motive, create a learning task, and collectively explore the theoretical content has the potential for developing students' relational agency (Edwards, 2005). However, learning activity is fragile, so to speak, and whether students establish a learning activity (Davydov, 2008) depends on several factors. For example, the subject-specific teaching situation in the form of a problem situation needs to highlight theoretical content and be designed to enable the students to perceive that there is a real problem to solve (Repkin, 2003). Furthermore, teaching based on the central principles of learning activity differs from much of the mathematics teaching in Swedish classrooms (Bråting et al., 2019; Hansson, 2011; Johansson, 2006; Larsson & Ryve, 2012).

These results allow reflection on what can promote and enhance young students' early algebraic thinking and the identified indicators exemplify what teachers can pay attention to when striving to develop students' algebraic thinking when working on algebraic expressions.

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