ON BEING STUCK ON A MATHEMATICAL PROBLEM:
WHAT DOES IT MEAN TO HAVE SOMETHING COME-TO-MIND?

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Abstract Everyone gets stuck sometimes, and it can sometimes be frustrating, even debilitating rather than stimulating. However, being stuck is an honourable and useful state because that is when it is possible to learn about mathematics, about mathematical thinking, and about oneself. This applies especially to teachers, because the best way to sensitise yourself to learners' struggles is to experience parallel struggles yourself. People are usually eager to get unstuck, to locate and enact some hopefully helpful action, without attending to how they got stuck in the first place, nor how the fresh action arose. I propose to dwell in the states of becoming and being stuck, and to use this as a springboard to examine and amplify the notion of 'having some possibility come-to-mind' as a means to get unstuck. This will include an expansion of the notions of *system 1* and *system 2* (automatic-habitual reaction and considered response) from dual systems theory, so as to take account of the full human psyche and development of the role of reflection. My method will be as phenomenological as possible, drawing on specific accounts from my own experience, but hoping to resonate with the experience of readers.

1 Introduction

Everyone gets stuck some times. I have often suggested that "being stuck is an honourable state" and that if you don't get stuck, you are unlikely to learn much about fostering and sustaining mathematical thinking in others, much less in yourself (Mason, Burton & Stacey 1982/2010). Put more sharply, "A solved problem is as useful to the mind as a broken sword on the battlefield" (Shah 1970, p. 142). Learning to persevere, to enjoy the state of hopeful not-yet-knowing but being on the edge of knowing, is valuable, while feeling stymied and hopeless, even helpless, is not so useful because once negative emotions kick in, the current state of feeling hopeless may transfer to a sense of oneself as hopeless, whether at doing this problem, at this topic, or even at mathematics altogether. What is a current state of 'can't yet' can all too easily turn into 'can't' and may be transferred into 'can't ever' which is manifested in the future as 'won't try'. Dweck (2000) reports research-based techniques for trying to persuade students to convert back to the language of "didn't" rather than "can't".

As soon as an action is enacted we are off and running, without attending to how we became mired in the first place. Often we don't even know where the fresh action came from: it has come-to-action but not necessarily to cognition, to conscious awareness. But as a teacher it is vital to be aware of how learners become stuck, and what sort of prompts will enable them not simply to get unstuck but to learn from the experience, instead of amplifying the desire to 'get unstuck at all costs'. It is necessary therefore as a teacher to uphold the state of frustration in order to sustain students' work. Here I want to dwell in

and on the state of being stuck, to consider briefly the psychology and sociology of that state in the light of various theories and commonplace clichés, and to elaborate on the role of reflection in developing a repertoire of mathematical actions which can inform future practice.

2 Method of Enquiry

My preference always is to work with lived experience, phenomenologically. That means generating immediate experience which may resonate (or dissonate) with other people's past experience. I am usually less interested in my own musings than in evoking awareness through invoking actions in those with whom I am working. However, in regard to 'getting stuck', there simply is not sufficient time to provide tasks that might get readers stuck in different ways, because different people will get stuck on different things in different situations. So my method here is to recall my own experience, and, through brief-but-vivid accounts and invitations to the reader to reflect on their own experience, to evoke memories of similar parallel experiences in my readers. It is readers' memories which constitute the 'data'. This way of working is consistent with what I call the *discipline of noticing* (Mason 2002).

Accounts of instances from my own experience will be followed by what they bring-to-mind in the way of psychological and sociological insights from the literature and from my experience. Thus I will not waste time adumbrating in advance various frameworks of distinctions which constitute my underlying theories, but rather let these emerge in the analytic considerations following reported incidents. Experiences generalise to phenomena, and these bring-to-mind, that is, resonate and trigger discourse from the literature which can be used to make distinctions and connections in and between experiences.

3 Being Stuck

In what follows I offer a sequence of incidents from my own experience, which may prompt personal reflection in the reader, followed by some analysis connecting observations to the literature.

3.1 Incident 1

When my son was about 8 or 9, he asked where I was going that day, and I told him I was going to Liverpool to give a talk. He asked what it was about so I told him that the title was "What to do when you get stuck". After a thoughtful silence he announced "Get out and push!".

This is extremely sage advice from one so young! The whole issue about being stuck is how to 'get out and push'. Here, 'get out' refers to becoming sufficiently aware of being stuck so that you can withdraw far enough from the current action(s) so as to change actions. 'Push' refers to actions that become available as a result of being released from being immersed in being stuck. Sometimes even though I am aware of being stuck, I may not have any alternative actions available, and even if a possible action becomes available, I

may not have sufficient will-power to initiate it. Becoming aware of being stuck is by no means inevitable, nor is it sufficient for getting unstuck, but it is a good start!

3.2 Reflection

Have you ever been stuck but only much later realised that you were in fact stuck?

Have you ever been stuck but found no fresh actions becoming available?

For example, have you ever been looking for something, finding yourself repeatedly looking in the same places, convinced that it must be there somewhere? Or have you ever carried out the same computation several times, often on a fresh piece of paper, vaguely hoping or expecting that the calculation will 'work out' (the way you hope or expect) this time? One state of being stuck is to be so immersed that I am blissfully unaware of it. Usually I am ever-hopeful that I am on the verge of making progress. I am unable to 'get out and push', that is, to invoke some new action.

There is an eastern teaching story that speaks to this condition, concerning someone who, late at night encounters a person searching for a lost key under a lamp post. After joining the search for some time, the first asks the second where the object was lost. "Ah, over there", came the response, pointing to the other side of the street where it is dark. "Then why are you looking here?", to which the reply was "There is more light over here". Continuing to pursue fruitless actions is a hallmark of the state of being immersed in being stuck.

A memorable story like this can act as a seed around which related incidents can accumulate. Through some combination of metaphoric resonance between the situation and other experiences you have had, and metonymic triggering through affective memory, a change of state of awareness can take place: becoming aware of being stuck. Perhaps even some related action (in this case, withdraw from the action and seek an alternative approach) can become available.

3.3 Reflection

Have you ever been stuck, with a vague sense of being stuck but without seeming to be able to do anything about it?

This has happened to me many times. I recall sitting at a desk in my master's year at college, struggling to complete an assignment by the next day, and repeatedly being vaguely aware of being stuck, of staring unfocused at the ceiling, of not knowing what to do, of not having any suitable action come to the fore, before eventually becoming sufficiently aware that I could in fact act by going and getting a drink, by standing up and walking about, or by stretching. I found that gazing at the ceiling was not effective, and that either I needed a narrower focus and concentration or else a release from inner tensions and tunnel vision. Sometimes letting go enables fresh neural networks to be activated!

In between the state of being sufficiently explicitly aware of being stuck so as to be able to take action to deal with it, and the state of being immersed or even lost in being stuck

there seems to be a whole spectrum of states with varying degrees of being vaguely aware of not making progress, of repeating actions already enacted, of not knowing what to do, and of no action taking place at all. What is most perplexing is how and when fresh actions do become available.

3.4 Analysis 1

There is a complex psychology here, because the hope or thought that progress is imminent can override admitting being stuck and so can block progress. Yet taking a positive rather than a negative stance may sometimes actually enable progress. Relaxing into not-knowing can allow fresh ideas to arise, whether after sleep, or during a period of enforced idleness such as being in the shower.

These states of being stuck can all be accounted-for in terms of whether sufficient energy is available so as to initiate some considered action, trying to go around the obstruction rather than repeatedly 'crashing on the rocks' by trying to barge through. We see this when watching a fly buzzing at a window, while a dog, faced with a window will seek a way around. Put another way, we can investigate and account-for being stuck in terms of whether there is sufficient energy to activate not only a fresh action, but a fresh perspective with enactive, affective, cognitive, attentive and intentional components.

Arising from awareness that people act, feel, think and attend differently in different situations, the human psyche can usefully be thought of as comprised of multiple adherences or coordinations between enaction, affect, cognition, attention and will. Thus a particular action may be strongly associated with particular emotions, specific trains of thought, sensitivities to notice, and energy to initiate action. Any one of these aspects, being activated, can bring the others to the fore. Each adherence is a particular interconnection between cognition, affect, enaction, attention and will, and each of these 'centres' itself has cognitive, affective, enactive and attentive components (Ouspensky 1950). For example, each adherence is associated with particular dispositions (an affective dimension) which itself comprises a particular relationship between cognition, affect, behaviour and will.

Such adherences have been described in terms of *selves*, as for example in Plato (Hamilton & Cairns 1961) and Bennett (1964); as *frames* Hudson (1968) and Minsky (1986); and as *micro-identities* or *virtual selves* Varela (1999), among others. Each 'self', 'frame' or adherence has characteristic triggers (as in Minsky's 'default parameters'). These are characteristic ways of transforming energy, characteristic ways of acting and thinking, characteristic sensitivities directing attention, and characteristic ways of taking initiative. The characteristic energy transformation of a particular self may be negative, positive or ambivalent. The ways of acting may be introverted or extroverted, socially sensitive or socially unaware, and so on. Different selves are seen in the Bhagavad Gita as different combinations of the three Gunas: *rajas* (initiative), *tamas* (receptiveness) and *sattva* (detachment), which together structure *Prakriti*, the realm of the seen (Ravindra 2009 p72). Getting unstuck can be seen as experiencing a shift of dominant adherence or a

rearrangement of collections of adherences so that fresh action is possible through a different combination of *rajas*, *tamas* and *sattva*, and hence of processing and transforming energy, leading to a change of state. Change of state is not always immediate, as the next incident illustrates.

3.5 Reflection

Can you recall struggling with some mathematical concept, finding it somehow elusive or incomplete, perhaps for a short time, but perhaps for an extended period?

I have vivid recall of several such incidents at high school and as an undergraduate, and many when attending mathematics seminars. The details are of course not important, as long as there is some recognition of struggling with some concept and gradually experiencing the fog lifting.

3.6 Incident 2A

I recall in my early years at the Open University being introduced to the function

$$f(x)=\begin{cases} x^{\lambda}\left\lfloor \dfrac{1}{x^{\mu}} \right\rfloor & x\neq 0 \\ 1 & x=0 \end{cases}$$ when $\lambda = \mu = 1$

as an example of a function which is continuous at 0 but has arbitrarily large slope arbitrarily close to 0. I gained familiarity with it by introducing the parameters λ and μ and exploring variations which are differentiable at 0 but which have arbitrarily large slope arbitrarily close to 0. By showing it to lots of people while rehearsing the reasoning, not only were my concept images for continuity and differentiability enriched, but my intuition about slopes actually changed.

3.7 Incident 2B

Working on the issue of $.\overline{9}$ with some third-year mathematics undergraduates at a prestigious university who were taking a mathematics education course, I recall vividly that one of them said "it may be 1 in analysis, but not out on the street", and this was agreed to by many of the students.

3.8 Analysis 2

The reflection together with these two incidents speak to ways in which familiarity and confidence concerning a complex or unfamiliar idea can develop, at various speeds, over time. Sometimes a fresh idea is grasped, assimilated, and internalised immediately, but most often there is a gradual development of a way of thinking through adopting, adapting, and internalising a way of acting. James (1890, 1925, p. 201) called this 'acting-as-if', which he saw as a very effective strategy for changing one's state. Incident 2A took place over several months. It relates to being stuck because an unexamined intuition (differentiability at a point must surely mean nearby points having nearby slopes?) is a form of rut in which it is possible to be trapped or stuck, quite unwittingly. Every teacher has encountered

students who resolutely stick to an earlier assumption despite being exposed to counter-examples. This is the action of *monster-barring* introduced by Lakatos (1976; see also MacHale 1980). Even when you have become aware of it, the educated intuition or insight may not become available in the moment when under pressure, allowing the naïve intuition to come to the surface and hold sway. Fischbein (1987) based his claim that intuitions are not displaced, merely overlaid, on this sort of experience displayed by learners.

The issue of $.\dot{9}$ is one that I have found engages undergraduates and high school students. It seems that there are deep seated concerns or questions (being stuck in a particular way of thinking about unfolding rather than completed infinities), and that over a period of time, as a teacher, at some point one becomes aware that rather than going through the motions when working with others, the fact that the value is 1 has been internalised and accepted. But this can sometimes take years. Of course the value is only 1 if you are working in the standard model of the reals, whereas in non-standard models, the value differs from 1 by an infinitesimal. This can provide a deep challenge, especially when the equality has finally been accepted, appreciated, internalised and apparently comprehended!

Gattegno (1987) used the term *awareness* to mean 'that which enables action', which might not be conscious (such as somatic actions like adjusting and maintaining breathing and heart rate). He spoke of 'educating awareness' meaning the process by which actions became internalised, and he proposed that 'only awareness is educable'. It seemed to me, drawing on common images from the Upanishads and elsewhere that this is in contrast to 'only behaviour is trainable' and 'only emotion is harnessable' (Mason 1992a) to which might be added, "only attention is orientable". Gattegno noticed that very often awareness is educated by "integration through subordination" (Gattegno 1987, 1990; see also Hewitt 1994; Young & Messum 2011), in which attention is deliberately drawn away from the action so that the action can be carried out in the future using only the minimum necessary attention to be carried through, calling upon the intellect only for occasional guidance when something unusual happens. This is typical of expert behaviour: the industrial slogan 'just in time' works for efficient action too: only sufficient minimal attention needed for an action need be focused on the action, freeing attention to be directed elsewhere. In some sense, a student is 'stuck' when they have to give full attention to the carrying out of an action.

A completely different discourse based on the biological metaphor of *assimilation* and *accommodation* was used by Piaget (1971) for much the same idea, though it led him to focus on *reflective abstraction* in which the learner is drawn out of immersion in action in order to become aware (consciously) of the action, with the intention of invoking it in the future. Building on a lecture by Bennett (1976) I articulated the *discipline of noticing* as, among other things, a collection of actions and practices designed to enhance the possibility of having a desirable action come to be enacted in the moment. Vygotsky (1978, 1981) spoke

of *internalising higher psychological processes* as being accessed first through the social by being in the presence of more experienced people manifesting useful actions, and then internalising these. It seems clear that each of these discourses adds a dimension of richness to the process of becoming familiar with, gaining facility with, and integrating into one's functioning, useful actions for getting oneself unstuck.

Incidents 2A and 2B illustrate a social aspect of coming to grips with concepts which challenge intuition, whether naïve or sophisticated. Participating in a community, whether as leader (teacher) or as participant can contribute to a gradual adoption of ways of thinking and acting. Bruner (1991 p4) asserted that "we organise our experience and our memories of human events mainly in the form of narrations". It is through narrative construction that we weave together the fragments of incidents, blending them into a single narrated experience. These narratives are the source of our sense of 'I', our supposed identity, even though, as Eastern mystics have maintained (Ravindra 2009) and as Norretranders (1998), McGilchrist (2009), Kahneman (2012) and others have shown, this is largely an illusion. Although James (1890 p224ff) talked about the 'stream of consciousness', it seems that most often not only do we recall experience in disconnected fragments which we try to glue together into a continuous narrative, but experience itself is fragmentary (Mason 1998, 2002). Even our memories of incidents are fictions, glued together from fragments so as to give ourselves a sense of coherence and unity. But trying to articulate to others can make a significant contribution to clarifying for oneself, to comprehending and appreciating connections and relationships. As the adage has it, 'the best way to learn is to teach' even if it is only oneself.

It can take a significant period of time during which one acts *as if* something is the case, eventually coming to accept and to internalise that acceptance so that future functioning takes it as being the case (James 1890 Vol 2 p321; 1925). Such internalised intuitions can be even more robust against further challenge than naïve intuitions. Unexamined assumptions can contribute to familiar actions not having the desired effect, and getting to grips with a new way of thinking can take time, a sensibility crystallized in the famous grook "Things Take Time" (Piet Hein's webref). The process is likely to be extended if habitual actions continue to be enacted rather than withdrawing from the action and considering alternatives. Throughout the period there may be times both of waxing and of waning confidence and clarity, all forms of being in transition, and a form of being stuck. The onion model for the growth of understanding which was elucidated by Pirie and Kieren (1989, 1994) nicely captures this ebb and flow.

Vergnaud (1983) used the term *theorem-in-action* to refer to a related phenomenon, in which students *act as if* they know a mathematical result, even though they have no explicit articulation. For example, young children often act as if they know that addition and multiplication are commutative; people accept that $\frac{2}{3} of \frac{3}{4}$ is the same as $\frac{3}{4} of \frac{2}{3}$ without being able to explain why, and often without even being aware that there is something to explain.

This is a subtle form of being stuck because you seem to be able to act, and yet you are 'stuck in a conceptual rut', confined by habits of thinking and acting, and dependent on the arising of internalised actions.

3.9 Incident 3

Early in my first year of high school (age 13), I recall being given a test in which one item consisted of four 4-digit numbers with the decimal point in different places. The task was to add them up. Having no idea what the dot referred to I aligned all the numbers, added, and then briefly pondered where to put the dot. I don't now recall where I put it, but I was definitely aware that I was unclear!

3.10 Analysis 3

When we don't know what to do, we do whatever we can do. We enact some familiar internalised action, which may or may not get processed through consciousness. For example, some students, when given a task, do or say the first thing that occurs to them, often without 'thinking'. Mathematics lessons provide an opportunity to work at responding thoughtfully rather than reacting automatically, to work at 'parking' initial impulses until they have been thought through or displaced by a better idea, and this could carry over into the awkward period of adolescence when choices are 'made' spontaneously and not always with much forethought or criticality.

Brown & van Lehn (1980) noticed that students are prone to enacting fragments of 'learned procedures'. Sometimes they may be uncertain as to what to do, as in incident 3, and sometimes they don't get as far as cognitive processing because a possible action is enacted and off they go in whatever direction it takes them. Van Lehn (1989) developed a comprehensive theory for how learned procedures develop bugs leading to both systematic and unsystematic errors in execution.

Bob Davis (1984) accounted for children adding numbers when working on a word problem in which they didn't know what else to do, by suggesting that they had spent a long time at first learning to add, then less time learning to subtract, even less to multiply and hardly any to divide. So naturally people turn to what is most familiar, most secure and confidence inspiring. This aligns with a framework called MGA (Manipulating–Getting-a-sense-of–Articulating) developed in the 1980s at the Open University (Mason & Johnston-Wilder 2004a, 2004b) which arose from a mixture of experience and the notion of spiral learning (Bruner 1960). We proposed that when you are stuck it is natural to turn to something confidence inspiring, like a familiar example, or a special case: something particular where you can act. The purpose of specialising, of manipulating the particular, is not to get an answer, but rather to see what is going on, to get a sense of underlying relationships. This 'sense' may then be articulated, and as the articulation gains in lucidity, succinctness and ease, it itself becomes a confidently manipulable entity, to be turned to in the future.

The incidents offered so far all indicate the difference between reacting and responding (Mason 2002, 2009), which is the essence of the dual system theory propounded by Kahenman (2002, 2012) and Kahneman & Frederick (2005), who drew on a distinction made by Stanovich and West (Kahneman 2012 p48, p450). It has been used to account for phenomena in mathematics education in Leron (undated) and Leron & Hazzan (1997). In the theory of dual processes, system 1 (S1) consists of automatic, habitual reactions which are enacted without cognitive processing (James 1890 p128-144). System 2 (S2) is a slower-functioning cognitive critical rationality. S1 provides S2 with conjectures based on whatever limited data is available, in line with past experience, a process characteristic of abduction (Eco 1983). It is often essential to act immediately, without considering pros and cons, but it is sometimes valuable, even essential, to respond rather than react, that is, to park initial impulses.

The etymological roots of *reacting* and *responding* are instructive, since *re-acting* refers to a cognitively unprocessed habitual act that is initiated without thought. When someone says or does something, other people may react automatically out of habit. Reactions may have some genetic basis, but are mostly what has been internalised and integrated into functioning, whether intentionally or through unrecognised enculturation such as through the use of language. Without habits we could not cope with the myriad of incoming stimuli, but at the same time, some habits are less than helpful and actually interfere with making a suitable response. By contrast, the etymological roots of *respond* are in the Italian verb *respondere* meaning 'able to justify', signalling involvement of the cognitive dimension of the psyche, spinning a story to account for actions already taken or at least prepared. To respond is to consider, to process possible actions through the intellect. Unfortunately in English we lack a term for specifically emotional automaticity, such as *re-emoting*: thinking and acting can be blocked by an emotional energy kicking in, often due to unrecognised metonymies with past experience which function on the surface (Lacan 1985) but below consciousness.

The difference between reacting and responding is critical in education because it can be used to account for a good deal of student behaviour: for example, when someone known to 'know' nevertheless makes a silly mistake, or when a state of being stuck arises because immediate habitual reactions lock students and teachers into an unhelpful line of thinking (see also Paz & Leron 2010 p36).

To illustrate system S1, Kahneman & Frederick (2005) use the following example:

3.11 Incident 4

A baseball bat and ball cost together one dollar and 10 cents. The bat costs one dollar more than the ball. How much does the ball cost?

This 'incident' is not so much about being stuck in not knowing what to do, but rather about reacting with a 'conjecture that requires modification', that is, being stuck in a rut of

inappropriate action. Almost everyone reports an initial tendency to answer '10 cents' because the sum $1.10 separates naturally into $1 and 10 cents.

3.12 Analysis 4

Frederick (quoted in Paz & Leron 2009, p. 23; and in Kahneman 2002, p. 451; see also Kahneman and Frederick 2005, p273) found that many intelligent people yield to this immediate impulse: 50% (47/93) of Princeton students and 56% (164/293) of students at the University of Michigan gave the wrong answer. Of course an impulse may serve as a useful starting conjecture requiring modification, but only if, in Kahenman's terms, S2 kicks in and delays the proposed S1 reaction, parking it while other options are considered.

Systems S1 and S2 correspond quite well to the common sense notions of intuitive and analytical thinking, and are in alignment with the ancient psychology of the Bhagavad Gita and the Upanishads where perception and cognition are recognised as operating at different speeds and in different ways. Kahneman (2012, p. 105) sees S2 as a kind of lazy monitor, fed proposals by S1 but disinclined to be critical. He uses the distinctions to urge a greater rationality based on educating intuitions concerning statistics and being apprised of the many ways in which S1 deceives us through actions such as priming (p52-58), availability and a range of other factors.

There is neurological evidence that these modes are activated by different parts of the brain and that they have different evolutionary origins. McGilchrist (2009) makes a similar distinction but associates them with the left and right sides of the brain. Kahneman sees S2 as evolutionarily more recent, largely reflecting cultural evolution. S1 acts originate somewhere between perception and (analytical) cognition. Taking S1-S2 seriously raises important consequences for how empirical findings in cognitive psychology are interpreted, including application to mathematics and computer science education research (Paz & Leron 2009 p23)

However, when a more refined model of the human psyche is used, drawing on aspects of ancient psychology, there are indications of a whole spectrum of modes, and of ways in which these modes interact. These arise from multiple adherences with different characteristics being activated at different times. As the next incident indicates, sometimes a self with the disposition to persevere is over-ruled by another self which wants to get unstuck and get on with something else, and because of stronger claims or stronger energy flow, it dominates temporarily.

4 Becoming Unstuck: Pushing & Pondering

Sometimes an action is merely part of being stuck in a rut, but sometimes a fresh action is enacted. Where do these fresh possibilities come from, and how might they be accounted for psychologically and sociologically?

4.1 Incident 5

On the look out for situations exploiting symmetry, I recently noticed the following result in the solutions to problems section of a journal (Dolan 2014):

$$\sum_{k=1}^{A-1}\left\lfloor \frac{kB}{A} \right\rfloor = \sum_{k=1}^{B-1}\left\lfloor \frac{kA}{B} \right\rfloor .$$

Try as I might I could not see how to prove it. I tried particular examples and I tried generalising from the detailed workings of particular examples; I became aware of a connection with the Euclidean algorithm; I drew a diagram that gave the result but which I could not see how to justify. Nothing 'worked'.

Eventually I resorted to looking at the solution (I had enough other problems I was working on and did not want to divert my attention!). I followed and reproduced the reasoning. But later, thinking through the reasoning while on a bus without access to the article, I got stuck again! I could not reproduce the elementary observation on which the reasoning depended. Partly it was because I could not remember 'the solution on the page', and partly because my unhelpful previous thinking kept coming back and getting in the way. I puzzled and puzzled, trying to reproduce what I had (thought I had) previously appreciated and comprehended.

The key idea which I kept missing is that $\lfloor km \rfloor$ counts the number of integer points above the x-axis and on or below the line $y = mx$ at $x = k$. Although I had tried to count lattice points myself it had been without success. Having written this paragraph, I went back to the problem and suddenly saw how it worked. I re-produced the reasoning for myself from scratch, and it feels as though I won't forget it in a hurry!

4.2 Analysis 5

For me this was a good example of being stuck in a rut, of one particular adherence with a fixed perception, attention structure and action dominating, trying to connect algebra to a diagram but thinking in an ineffective way. My attention was focusing on what proved to be unfruitful. It kept returning in the absence of any other ideas coming-to-mind, like the person searching under the light.

In order to learn from the experience it was necessary to reflect back and to consider what might have been effective actions. I did try examples, but only arithmetically; when I tried to use a diagram, I did not hit on an effective way of displaying the problem. To get out of a rut, it may not be sufficient to 'get out and push': sometimes you have to 'get out and ponder', waiting for inspiration, clearing your mind so as to be prepared for a fresh juxtaposition, as Hadamard (1945) describes. The biblical parable of the wise and foolish virgins comes to mind from Matthew 25: 1-13. Being prepared means not being fully immersed. It means having an inner witness or observer (Mason *et al* 1982/2010), what Schoenfeld (1985) called an 'inner executive'. This could be seen as the role of S2, or of different adherences, but it is also useful to see this as an extra dimension of the psyche. The witness is what is available to be awake to unexpected situations, connections and

insights. The richer the repertoire of successful actions, the more likely it is that something useful will come-to-action, come-to-emotion and come-to-intellect.

Frustrated, I turned to help from outside, but found that that 'help' was not sufficiently robust to inform my future activity. I subsided back into the rut.

In order to learn from the experience it is necessary to lodge the experience of being stuck and the awareness that released suitable actions into memory, so that it is accessible to future metaphoric resonance and-or metonymic triggering. Labelling certainly helps. As more instances are attached to the same label, there is a growing richness for resonance and triggering.

5 Reflection, Post-Paration and Pre-Paration

Getting stuck is not in itself a recipe for learning, nor indeed is a succession of getting stuck and then getting unstuck again. At one extreme, when you are stuck you can consult a list of heuristics (I have seen as many as 99 assembled in one place) and somehow choose one that might be of help in your situation. At the other extreme, some action 'works' and you find yourself unstuck and forging ahead, without trying to learn from the experience. As I have said many times (Mason 1992)

> *One thing we don't seem to learn from experience, is that we don't often learn from experience alone.*

Something else is required. Worse,

> *A succession of experiences does not add up to an experience of succession.*

This turns out to be a version of an assertion by James (1892, p. 628) that a succession of feelings does not add up to a feeling of succession. In other words, more is required than simply a succession of experiences if we are going to learn from them.

Schön (1983) distinguished between *reflection in action* and *reflection on action*. He used the former to mean being sensitive to possibilities in the moment rather than being driven entirely by habits and automated-internalised action. But it could also mean an inner witness which is able to observe and comment without being embroiled in the action. The Rg Veda has a stanza which speaks to this:

> *Two birds, close-yoked companions, cling to the self-same tree.*
> *Of these one eats of the sweet fruit,*
> *The other, nothing eating, looks on intent. (Zaehner 1966, p. 210)*

Sometimes interpreted concerning mortality and immortality, the stanza can also be read as a description of a state sometimes referred to as *cosmic consciousness* (Burke 1905), of *mindfulness* (Langer 1997), or *self-observation* (Ouspensky 1950). The second bird is a witness or observer which can ask questions (why are we doing this?) but does not itself initiate action associated with the specific activity. But it can awaken awareness of being lost or immersed in being stuck.

Reflection-on-action comes later, in the post-paration phase (as distinct from pre-paration, and from *paration* (literally, put aside) which I use to mean *noticing in the*

moment, as in 'reflection-in-action'). Post-paration can be turned into pre-paration through imagining yourself in some future situation as vividly and completely as possible, enacting some action which helped you get unstuck this time (Mason 2002). This is how experience can inform learning.

When a student is stuck and a teacher intervenes, there are all sorts of possibilities. John Holt (1964 p24-25) describes how Ruth takes him for a ride, biding her time until he asks a really simple question that she can answer, but with no idea of how that relates to the original difficulty. Bauersfeld (1995) used the term *funnelling* for a sequence of questions which are ever more explicit, specific and pointed until the student can give a response without thinking. Consequently the student is unlikely to be aware of the origins of the teachers' questions, so there is no cognised sense of a sequence of actions that the student might use for themselves in the future. Leron & Hazzan (1979) provide a possible inner monologue of a student in a funnelling situation, in which the students' attention is as much on the process of the question sequence as on the questions themselves. Indeed the teacher's attention is likely to be fixed on the desired outcome and on triggers or cues towards that. Student behaviour may imply that the student is coping with the situation, trying to make sense and trying to meet expectations, but not being helped to formulate a narrative. Coping with the situation (which includes assumptions by the student of what the teacher expects) seems to amplify spontaneous reactions (or else silence) rather than promoting thoughtful, considered response.

It is through withdrawing from action and reflecting on that action that experiences of successful actions can be associated with particular features of being stuck. Just as the story of the lost key can act as a seed for the accumulation of related experiences, so too an incident can act in the same way, accumulating related experiences and associated actions. This enriches the possibilities for metonymic triggering from and metaphoric resonance with past experience, leading to awareness of being stuck, and access to fresh actions.

Nelissen and Tomic (1996) review some 20 years of Russian research into the nature and development of reflection. They see reflection as inner dialogue and as such, as an instance of a 'higher psychological process' in the sense of Vygotsky (*op cit*). Through dialog people find their own ideas challenged by others, and learn to challenge themselves. But reflection is also the process by which scientific concepts emerge, what Gattegno (*op cit*) referred to as 'educating awareness'. Davydov (1990) challenges the classic Western sequence of 'enactive–iconic–symbolic' in which ideas are met first with material objects, then diagrams and images, and then symbolic expressions. Rather, Davydov proposes that conceptualisation comes about most effectively when 'well established ideas (ideal models that can be seen as propositional representations) are used to understand reality' (Nelissen & Tomic 1996 p39). Later researchers have blurred the distinction and found it more useful to see the empirical and the theoretical as co-emergent. In a range of experiments with young children, evidence has been garnered to support the notion that young children can

display reflective behaviour (explicit awareness of actions) and that children can be enculturated into a reflective stance, with good results (Nelissen & Tomic 1996 p39-41).

There is considerable evidence from post-Davydovian projects that promoting self-narrative contributes to learning such as Schmittau (2004) and Dougherty (2008). Others, such as Chi (1989); Chi, Bassok, Lewis, Reiman & Glasser (1998); and Chi, de Leeuw, Chiu, & LaVancher (1994) have developed a similar idea independently. Self-explanation has been applied explicitly and to good effect in mathematics education in various places, notably Hodds, Alcock & Inglis (2014).

Further outcomes of the Russian research have led to distinguishing between 'personal reflection' which involves formulating meaning to their own actions, and 'intellectual reflection' which focuses on inner questions about the nature of the problem at hand, typically, of the type promoted by Pólya (1962). *Personal reflection* is itself seen as taking three forms: situational, pertaining to the particular situation; retrospective; and perspective-generating. The latter arises particularly sharply when there is some conflict, some dissonance between effect and desire, such as the ineffectiveness of some proposed strategy (Nelissen & Tomic 1996 p46). Finally, Russian researchers distinguished between 'productive' and 'reproductive' reflection. The former involves high levels of intellectual and perspective-seeking personal reflection; the latter is more like the spectrum of 'being stuck but without a specific action being enacted' as discussed previously.

6 Coming-to-Mind: Action, emotion, intellect, attention, will and witness

The cliché or stock phrase in English for getting unstuck is 'coming-to-mind' or 'having something come-to-mind'. For example Kahneman & Frederick (2005) use the phrase to mean an action that stems from the automaticities of the speaking part of the brain:

> *The effect of concurrent cognitive tasks provides the most useful indication of whether a given mental process belongs to system 1 or system 2. Because the overall capacity for mental effort is limited, effortful processes tend to disrupt each other, whereas effortless processes neither cause nor suffer much interference when combined with other tasks [...] People who are occupied by a demanding mental activity [...] are much more likely to respond to another task by blurting out whatever comes to mind [...]. (Kahnemann & Frederick 2005, p. 268)*

It seems that the term *mind* is used by many people to encompass a variety of aspects of the human psyche. Relatively recently neuroscientists have been adjusting their theories of mind to acknowledge what has been known for centuries in the East, that the intellect does not always initiate action. Rather action and perception occur together, with action, particularly re-action, preceding conscious thought. For example, Guillery (2014), based on Sherman & Guillery (2013) proposes a reinterpretation of the thalamus as a gateway or bifurcation point where instructions to motor neurons (to act) are copied to the cortex: it is not the cortex that initiates the signal to the motor-neurons. Norretranders (1998) cites

neuro-scientific evidence that the assumption that consciousness is the generator and director of action is an illusion. Rather, consciousness is a post-facto phenomenon. Mandler (1989) proposed what was well known in Eastern psychology, that especially under duress, the body reacts first, emotions become aroused next, and thoughts in the intellect-cognition emerge later, a slow third. This can be verified by close observation of oneself, but most clearly when there is a sudden noise: the body acts, then energy starts to flow in and through emotions before any coherent account is available to the intellect. From Eastern psychology also comes the notion that human beings construct narratives which help to self-calm (Ouspensky 1950), in order to maintain the user-illusion that the intellect is in charge. This aligns perfectly with the notion promoted by Bruner (*op cit*) of human beings as narrative animals.

If 'mind' is reconfigured to consist of action, emotion, intellect, attention, will and witness, then reaction refers to the activation of the characteristic functionings of a particular adherence or self, while response refers to engagement of one or more adherences or selves with all aspects fully contributing to the direction of action, including will.

Any or all of action, affect, cognition, attention, will and the witness can come alive automatically, through habit, through well established pathways, in S1. They can also contribute to triggering the functioning of S2, to intentional, participative involvement in considering possibilities. It may be more informative therefore to see S2 as a spectrum of different adherences or selves contributing to the 'common presence' of the individual within the socio-cultural-historical norms into which they have been enculturated.

If we took our time over every action we would of course stop breathing, and even at the macro level we would instantly become dysfunctional. But it is also important to continue to question the efficacy of habits, to be in a state of being challenged about the actions we enact, the emotions we evince, the thoughts and ways of perceiving that we experience and the ways of attending that have become second nature.

In terms of being stuck and getting unstuck, in order for automaticities to get us unstuck it is necessary to have developed a repertoire of actions and of experience of successful use of those actions so that possibilities come to action, to heart, to head and to will. But in order to be creative, to experience insight, and to make connections between apparently disparate situations, it is valuable to foster mindfulness in which we participate fully in making choices by considering possibilities and consequences before acting, before emoting, and before launching into customary patterns of thinking, perceiving and initiating.

The notion of mindfulness (Langer 1989) has been developed recently into a Western approach treating depression and other clinical ailments, but also for general living (Williams & Penman 2011). Of course this is of no surprise to anyone acquainted with Buddhist teachings. Here there is a state beyond ordinary consciousness, when an inner

witness is present, observing, but not engaging (Mason 2002). Roots for this go back as far as written records, as in the stanza from the Rg Veda quoted earlier.

The other lesser-known aspect of coming-to-mind, namely coming-to-will deserves more attention. How is it that I can be aware that an action is required, even be consciously aware of what is required, and still not enact that action? Examples abound, at least in my own experience. It is as if there is an energy threshold which needs to be exceeded before action is initiated. Note that this is in complete contrast to re-action which is immediate and unconsidered.

7 Social Aspects of Being Stuck

The current fad of recommending collaborative group work is based on the observation that often when people work together they can feed off each other's ideas so that collectively they can achieve more than can any individual. Certainly when you are stuck it can be very helpful to seek out a colleague and to try to articulate your current thinking: where you are stuck and perhaps even why.

Incident 6

> *Stuck on a problem I go down the corridor and ask a colleague if I can explain my problem. The colleague sits there, doing nothing beyond appearing to give me some attention. At the end, I thank them and return to my office with renewed vigour and a fresh view of my problem.*

The mere act of trying to articulate can bring clarity. But this is not usually what happens when people work in groups.

Many people require some private time in order to internalise the task or problem, and to allow some things to come not just to action but to cognition, so that they can be considered. Only then does it make sense to engage in collective discussion and idea generation. Even so, group work requires individuals to edit their own thoughts, and suppress their own ideas so as to be able to listen carefully and considerately to others. Having generated ideas, a period of personal work, or perhaps paired-work may be needed in order to consider ramifications of all the suggestions. Continuing in a group may lead to the loudest or quickest taking charge while others struggle to keep up or drift off. Every time someone fresh offers a comment, everyone else has to suspend their own thinking in order to take on board the new contribution.

When a group gets stuck, with no fresh ideas being introduced but perhaps old ideas being re-circulated (again, the story of searching for the key mentioned earlier surfaces) it is time to return to individual work (or work in pairs) to wait for ideas to settle and for something fresh to appear.

8 Conclusion

Being stuck is a state in which either there are no actions to carry out, nothing to do, or where there are actions, but these are either repetitions of previously unsuccessful actions,

or the working out of habitual acts, which mean that situations proceed as normal: you are stuck in a rut. Through inspecting experience it seems that there is a spectrum of states of being stuck from being blissfully unaware, through being 'sort of' aware but unable to act, to being explicitly aware.

You can be stuck in a rut, endlessly going over the same thoughts, trying the same actions, without success. In eastern traditions this is sometimes referred to as the *chattering monkeys*, or in the West, the *broken record*: a habit keeps being enacted, a thought keeps recurring, a worry keeps surfacing. You keep looking under the light rather than in a relevant place or in a pertinent manner.

To get unstuck, you need some act to enact, to come-to-action, but if it is based on habit, unprocessed by cognition or thought, it may not actually be helpful, hence the need for the development of an inner witness, of mindfulness, of a monitor or executive which brings possible actions and nascent emotions to cognition before actions are enacted or released.

Past experience associated with failure may trigger emotional response which may then block possibilities of cognition or will, or, conversely, may trigger a desire to comprehend, producing a burst of energy to take initiative.

A repertoire of possible acts is necessary so that there are actions on which to fall back. These actions may be associated with particular mathematical concepts, with the use of mathematically oriented natural powers, or with themes that pervade mathematics. But actions alone are not enough. The development of a mathematical witness or monitor is invaluable: something that is not caught up in the action and so can draw attention away from the action, enabling more reflective, contemplative thoughts coupled with appropriate feelings. These form what might be thought of as mathematical adherences or selves who can take over control, who can process energies in mathematically useful ways, feeding the will so that initiative can be taken to persevere, whether by pausing and contemplating, or by pursuing some fresh direction.

An obvious conclusion, in alignment with experience at the Open University over several years, is that before dropping learners into unfamiliar 'problems' it is vital to give them carefully designed tasks which invoke mathematical actions that are going to be useful in the future. For example, imagining and expressing; specialising (trying examples) & generalising; conjecturing & convincing; being systematic and being adventurous; working forwards and working backwards, and so on. When some or most learners have used such an action naturally, their attention can be drawn to it, a label can be provided or negotiated, and then that label can be used to prompt use of that action in the future. Over time, the prompts can be made decreasingly explicit, using prompts like "what question am I going to ask you?" in a process of scaffolding and fading (Seeley Brown, Duguid and Collins 1989; Love & Mason 1992) until learners have internalised the action and developed from initiating it for themselves (consciously, attentively, intentionally) to experiencing it enacted without their explicit volition. They can then be said to have internalised a higher

psychological process (Vygotsky), educated an awareness (Gattegno); accommodated (Piaget), integrated an action into their repertoire of functionings.

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