

## **PROBLEM SOLVING AS AN INSTRUCTIONAL METHOD: THE USE OF OPEN PROBLEMS IN TECHNOLOGY PROBLEM SOLVING INSTRUCTION**

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**Abstract** Problem solving is not only an instructional goal, but also an instructional method. As an instructional method it can be used to build new mathematical knowledge, to solve problems that arise in mathematics and in other contexts, to apply and adapt a variety of problem-solving strategies, and to monitor and reflect on the mathematical problem-solving processes. However, depicting complexity of thinking and learning processes in such environments offers challenges to researchers. A possible solution may be through multiple perspective. On one exemplary problem this instructional method will be demonstrated in a technological context including then behaviors, dispositions and knowledge observed as a result of problem solving investigations in a technological context. These are discussed from three different perspectives – students', lecturer's and researcher's offering a rich portrait of a problem solving mathematical activity in a technological context. Implications for mathematics instruction at the secondary and tertiary level will be given at the end of report.

### **1 Introduction: Problem solving in mathematics curricula**

Results from PISA and TIMS study on mathematical achievements and rapid mathematization in social and work areas dictated changes in mathematics curricula, and schooling practices (Fey, Hollenbeck, & Wray, 2010). Nowadays, topics taught in mathematical classes require more than mere arithmetic or calculation skills, but rather extension and adaptability of previous knowledge, and flexibility in thinking. This vision can be achieved with problem solving having a greater role in mathematics instruction. Already in the 80s NCTM recommended “problem solving be the focus of school mathematics” (1980, p. 1), later endorsing that problem solving to be a main focus of school mathematics (2000). In 2000 NCTM stated:

*Solving problems is not only a goal of learning mathematics but also a major means of doing so. By learning problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations. (p. 52)*

These changes are recognizable worldwide – Germany, USA, Finland, Austria, to name a few – which set goal of problem solving as an instructional goal. Hence, the goals for school mathematics is to have teachers prepare students for challenges of a new technological world by becoming mathematical problem solvers through developing their own ability to think mathematically, critically, flexibly, and acquire mathematical power (the ability to do and have insight into learning mathematics).

Another change in school mathematics includes the use of different technological tools at all levels. More specifically, dynamic geometry software (DGS<sup>1</sup>) are advocated emphasizing its nature helping engaging students in meaningful mathematical activities, promotes deeper understanding of concepts and problem solving. Wilson, Fernandez and Hadaway (1993) recognized the role of DGS as a tool for mathematics problem solving, which supports processes such as pattern recognizing, conjecturing, testing, rejecting and refining conjectures, generalizing, and abstracting. Moreover, DGS interactive nature may help in engaging students in meaningful mathematical activities and promotes deeper understanding (Lee & Hollebrands 2008; Zbiek, Heid, Blume, & Dick, 2007), adding also new possibilities when problem solving.

Nowadays this picture is bipolar; while many countries redesigned their curriculum to fit the PISA and NCTM recommendations, these impacts are either to some extent seen in teaching or not at all (Reiss & Törner, 2007). Preservice and inservice teachers often provide rationale for not incorporating problem solving activities in school mathematics. These include arguments, such as time, overwhelming school content curriculum, and missing competence to implement and manage a problem-oriented classroom. In this article I focus on learning gains as a result of performance on different problem solving investigations with DGS in a problem solving course embedded in a technological context. In what follows, I review literature that informed the design of the course. I then describe the larger context in which study was situated – a researcher-student-lecturer partnership – that was attempting to better understand the problem solving activity in technological problem solving course. Next, I describe a description of the methods before reporting the findings, which focus on learning gains from the above three mentioned perspectives. I conclude by considering implications of the study results for school and university mathematics courses.

## 2 Theoretical considerations

### 2.1 Problem solving as an instructional method in a technological context

Problem solving is not only an instructional goal, but also an instructional method. Schoen (2003) described this instructional approach:

*As students attempt to solve rich problem tasks, they come to understand the mathematical concepts and methods involved, become more adept at mathematical problem solving, and develop mathematical habits of mind that are useful ways to think about any mathematical situation. (p. xi)*

One possible and widely accepted way to achieve this goal is through the open approach, where students engage in open, practical and investigative tasks (Pehkonen, 1992, 2001).

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<sup>1</sup> The name describes in its name the main features of the software: direct manipulation of geometric figures possible via a pointing device, for instance, by dragging parts of the figure.

Open problems are attractive because of their nature; they allow students with the opportunities to generate several approaches and/or different correct solutions placing very little constraints on the students' methods, to share and discuss their thinking, and to make decisions and justify them. Its effective use can foster higher-order thinking and promote development of mathematical habits of mind. According to Cuoco, Goldenberg and Mark (1996), habits of mind are the ways of thinking that are there to allow students to develop a myriad of approaches and strategies that can be applicable in situations varying from challenges in school to those in life. Some of the habits of mind they discuss include but are not limited to looking for patterns, exploring, communicating, arguing, conjecturing, refuting, and generalizing. The habits are there to enlighten students about the creation of mathematics, and most importantly to help them learn the way mathematicians think about mathematics. Consequently, students' engagement in these habits helps develop and increase their ability to determining on their own how to think mathematically. Open problems differ in their nature: the starting situation, the goal situation or the problem solving approach can be open. Thus, with each type of problem students have the opportunity to show their creativity and engage in different mathematical habits of mind. In this paper I focus on problem on strategy-open problems. As the name says, the start and the goal situation are exactly given, however, the task allows for many approaches to achieve the goals situation. Such problems are suitable for mathematics teaching at different levels of mathematics education, and can be therefore posed multiple times allowing the use of different mathematics and heuristics (Pehkonen, 2001).

The implementation of DGS when open problem solving creates an opportunity to transform a mathematics classroom into an environment of investigation of interesting phenomena where students engage in observing, manipulating, predicting, conjecturing, testing, and developing explanations for observed phenomena (NCTM, 2000, 2005). Goldenberg et al. (1988) argue that providing the opportunities and dynamic tools for students explorations promote the habits of mind, which I outline above. Zbiek et al. (2007) contend that student engagement in such activities using technology tools allows students their personal problem solving experience through habits of mind (e.g., pattern recognition, conjecturing, generalizing, abstracting) they engage in. Hence, DGSs provide the user a well-tuned system within which different mathematical concepts and mathematical problems may be explored (Hoyles & Noss, 2003). On the down side, users may tend to abuse the power of DGS. Instead of appreciation for the structure of the system, the user uses the system because he or she wants to get the problem done (e.g., using DGS as a verification tool) which takes away the cognitive load of mathematical thinking (Hoyles & Noss, 2003; Olive & Makar, 2010). In the following section I discuss from a theoretical basis how DGS may influence student problem solving in mathematics.

## 2.2 Different technological effects during problem solving

Technology supports doing mathematics. It is a lens for extending what mathematics we can do and how we approach our mathematics (Wilson et al., 1993). The relationship between the technology and the user itself can be observed in terms of (a) effects *with* technology (how use of technology often enhances intellectual performance; effects of technology), (b) effects *of* technology (how using a technology may leave cognitive residues that enhance performance even without the technology), and (c) effects *through* technology (how technology sometimes does not just enhance performance but fundamentally reorganizes it) (Salomon and Perkins, 2005).

Effects *with* technology occur when certain intellectual operations are exclusively carried about by the technology, dividing the labor between user and technology. Thus, technology is used to improve intellectual performance while one is operating the tool (e.g. intellectual partnership). DGS's affordances (see Sect. 1) provide the user with the opportunities of intellectual partnership. In the mathematics classroom, students use DGS to perform computations involving large or tedious numbers, to construct objects and figures involving tedious construction steps or to graph complicated functions (e.g., complex cognitive processing). These behaviours in conjunction with solving a given problem exemplify effects with technology. Yet, as a result of these experiences some cognitive capabilities can develop that remain available without the tool – so called, complementary theme of “effects of technology” –, which is discussed in the following.

Effects *of* technology are those that leave cognitive residues, positive or negative, that persist when the technology is no longer in use, and after a longer period of using it. Salomon and Perkins (2005) described the manifestation of this type of effect as the acquisition of a new skill or the improved mastery (or atrophy) of an existing skill. In the mathematics classroom, DGS allows students the opportunity to test a vast number of geometric cases or conjectures very quickly. Whereas students who have not used such software packages like this may have trouble envisioning more than a few cases, those who have used the software may be able to envision many, even without the use of the software (Wilson et al., 1993). The problem solver can profit from such abilities in numerous instances, particularly when working on his or on her own, allowing them to master their thinking, problem solving skills and strategies, and to engage in in higher order activities in subsequent partnerships with cognitive tools. Transfer is, however, dependent on mindful abstraction of procedures, strategies, and self-regulation.

Effects *through* technology occur when use of the technology qualitatively, fundamentally restructures and reorganizes the activity system in question rather than just augmenting it. Salomon and Perkins (2005) stated such effects do not emerge quickly, but gradually emerge, because new technologies are often assimilated into already established systems where the technologies cannot be used to their fullest potential. With respect to mathematics classroom, the availability of DGS allows problem solvers to conjecture before generalizing, to experiment before proving, to “better understand the problem” before an

approach is chosen or a final statement is formulated, to “doubt” or “believe” in results given by the tool, etc. Thus, DGS aids in reshaping the typical structure of geometry teaching and problem solving from one that began with the formulization of the result and ended with visualization and analysis to another where that process is reversed (Herrera, Preiss, & Riera 2008).

In this study, the three-way framework of Salomon and Perkins (2005) was used to depict different technological effects on the cognitive enterprise when problem solving in DGS.

### **2.3 Understanding problem solving in a technological context through multiple perspectives**

Problem solving in a technological environment is a complex phenomenon involving many different protagonists: student, instructor, problem solving activity and technology. According to Maturana (1988) to make sense of an environment we should analyze it in the terms of perspectives and the theories we bring. According to him (1987), an objective observation of a phenomenon is not possible without examining individuals who take part of the realm in which the phenomenon takes place. In other words, we can not take a position from which we can objectively report. However, we have the opportunity to participate as observers to observe the observations of others and to have them watch us. Through discussions, we see the different aspects of a particular environment. In this way not only a vision, but also a comprehensive portrait of the mathematical training culture is built. Hence, one should observe the others, the others should observe us, and through a together communication we can see multiple perspective of a learning environment. Through it, we can create a rich portrait of a mathematical activity, which is being investigated, that under one lens observation would have not been possible. For that reason, to analyze the learning process in a technological problem solving instruction three perspectives were in focus: the perspective of a lecturer himself, the perspective of preservice and inservice teachers, and the perspective of the researcher.

## **3 Research questions**

In this paper I present one strategy-open problem, and discuss the learning gains with respect to problem solving in a technological context from three different perspectives. Starting from the above outlined theoretical considerations and empirical results the following research question with its accompanying sub-questions are of interest in this report:

- What learning gains with respect to problem solving can be observed in a technology problem solving course?
  - What benefits to problem solving do the pre- and inservice teachers observe as a result of technology problem solving instruction?
  - What patterns of behavior does the course instructor observe in the pre- and inservice teachers when engaged in technology problem solving?

- What effects on the problem solving processes of pre- and inservice teachers are observable in a technology problem solving instruction?

## **4 Methodology**

In this section I outline the qualitative design of the study, and introduce the participants of the study (pre- and inservice teachers, lecturer). In addition, I discuss in more detail the context of the course, and portray on one example type of problems used in the course with its possible solutions. At the end of the chapter, I introduce the data instruments and how these got analyzed to answer the research question with its accompanying sub-questions.

### **4.1 Setting and participants**

For this study, a qualitative research design was chosen, where one case was in focus, namely the course itself. In total 51 preservice and inservice teachers participated in one-semester long course, which was a part of undergraduate and graduate degree in mathematics education, respectively. The course took place once per week for a period of 2 hours. The course concentrated on using different software to solve mathematical problems, to demonstrate different problems and discuss pedagogical considerations. The course emphasized the exploration of mathematics problems from different contexts, extend problems, to pose problems, and to communicate mathematical demonstrations. The goal was that the participant become familiar with and operational with using technology tools to solve mathematics problems and construct new mathematical ideas using application software, to engage in open exploration, and to communicate the mathematical ideas that arise from applications software. During the course the participants got to choose tasks they wanted to explore, solve and write a report on. At the end of the course they had solve 13 open problems and submit a portfolio comprising of 13 research booklets (problem-solving protocol and a post-reflection protocol about problem solving).

### **4.2 Data collection**

Data collection methods for this study consisted of the following: (1) interviews, (2) videotaping of the course, (3) document review (of participant's solutions), and (4) the researcher's field notes. During the semester the participants were asked to keep a booklet comprising a problem-solving protocol and a post-reflection protocol about benefits of technology during this process and problem solving to allow for active engagement in knowledge construction. At the end of the semester, the students submitted 13 booklets in the form of a portfolio. In addition, they had to write 1-2 page reflection papers designed to encourage them to relate what they were learning in class to their own practice or experience. In particular, they had to reflect on the semester-long experience of problem solving in a technology environment, both through the eyes of a student and a future practitioner. For this paper, one particular problem was in focus (see Sect. 4.3). I interviewed the lecturer of the course after the course participants solved this problem. The

interview focused on gaining insights into the lecturer's perspective as an experienced technological problem solver with respect to his observations about his students' problem solving activity. In addition, I interviewed several students at the end of the semester, who were willing to participate in a 20' interview (n=7) to gain more insight into their perceptions of the importance of DGS to their problem solving when faced with open problems. I also kept field notes based on informal discussion I conducted with the participants in class each week. As students worked independently at their computer stations, I talked to them about his or her progress in class. To preserve my neutrality as an observer, I refrained from commenting on the quality of either their mathematical work or their written work throughout the semester.

### 4.3 Instrument: Strategy open problem

The following problem related to Gamow from 1947 was given to preservice and inservice teachers found in Libeskind (2008), and is in focus in this report.

**Gamow's hidden treasure.** Among his great-grandfather's papers, Marco found a parchment describing the location of a pirate treasure buried on a deserted island. The island contained a pine tree, an oak tree, and a gallows where traitors were hung. The parchment was accompanied by the following directions:

*Walk from the gallows to the pine tree, counting the number of steps. At the pine tree, turn  $90^\circ$  to the right. Walk the same distance and put a spike in the ground. Return to the gallows and walk to the oak tree, counting your steps. At the oak tree, turn  $90^\circ$  to the left, walk the same number of steps, and put another spike in the ground. The treasure is halfway between the spikes.*

Marco found the island and the two trees but could not find no trace of the gallows or the spikes, as both had probably rotted. In desperation, he began to dig at random but soon gave up because the island was too large. Your quest is to devise a plan to find the exact location of the treasure.

The problem was nonroutine; it required participants to use information and strategies in unfamiliar ways; that is, they demand strategy flexibility; thinking flexibility, such as logical thinking; abstract thinking; and transfer of mathematical knowledge to unfamiliar situations, as well as extension of previous knowledge and concepts. Its nature encouraged the participants to use the DGS to explore different mathematical ideas. The problem challenged the students to experiment, conjecture, and prove, if possible, and invited different strategies, problem solving paths, and extending existing knowledge and problems to new problems.

In the following sections I present the four possible solutions on the island treasure problem as developed by the participants using coordinate geometry, Euclidean geometry, vector algebra and complex numbers. Before solving the problem, the students were encouraged to use a type of interactive geometry software (Geometers' Sketchpad [GSP],

GeoGebra, Cinderella). Points P, O and G representing pine tree, oak tree, and gallows, respectively, are draggable. Point T representing the treasure cannot be dragged. Since points P and O are fixed, the students dragged point G, though its position is unknown. By dragging the point G the students discovered that no matter where the gallows located is, the location of the treasure is invariant. Dragging points P and O further strengthened this conjecture. While the use of DGS was crucial for formulating the conjecture, for proving the conjecture its use was not necessarily pivotal. Figure 1 is a representation of different students' solutions to show the invariance of the treasure location.

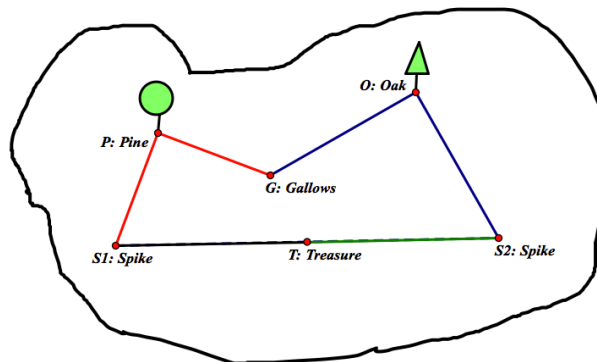


Figure 1 Finding location of the treasure

#### 4.4 Synthesis of the problem solving space

##### Solution using coordinate geometry

Without the loss of generality, a suitable coordinate system was chosen, such that points P and O are lying on the x-axis, and such that its midpoint coincides with the origin of the coordinate system. That said, the following coordinates were assigned: P (-p, 0), O (p, 0), and G (x, y). Further analysis of the initial situation prompted the discovery of triangles  $\Delta APS_1$  and  $\Delta GBP$ , and  $\Delta BGO$  and  $\Delta OCS_2$  (see Figure 2). For instance, segments  $PS_1$  and  $PG$  are congruent by construction. Similarly, triangles  $\Delta APS_1$  and  $\Delta GBP$  are right-angled triangles by construction. The angle analysis revealed that:  $m\angle BPG + m\angle PGB = 90^\circ$  and  $m\angle BPG + m\angle APS_1 = 90^\circ$ . Hence,  $m\angle PGB = m\angle APS_1$ . Thus by AAS theorem, are triangles  $\Delta APS_1$  and  $\Delta GBP$  congruent. This gives that  $AP = BG$  and  $AS_1 = PB$ . Analogously, can be shown why the triangles  $\Delta BGO$  and  $\Delta OCS_2$  are congruent. Hence, points  $S_1$  and  $S_2$  can be

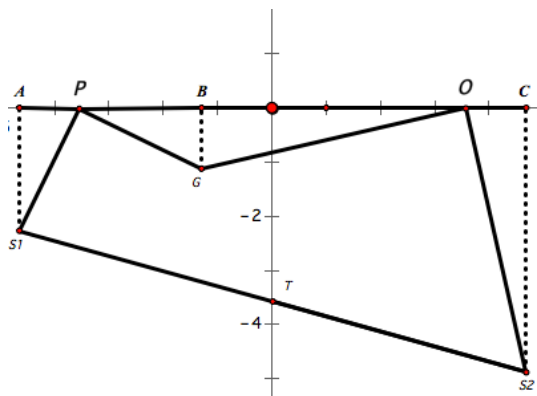


Figure 2 Coordinatization of the treasure island problem



easily coordinated;  $S_1 (y-p, -x-p)$  and  $S_2 (-y+p, x-p)$ . To coordinate point T, the midpoint formula was used giving T (o, -p). Hence, this showed that the location of the treasure is independent of the location (coordinates) of the gallows (point G).

**Solution using Euclidean geometry**

Let M be the midpoint of segment PO. When using the Euclidean geometry, the students again analyzed the given situation and deduced that triangles  $\Delta APS_1$  and  $\Delta GBP$ , and  $\Delta BGO$  and  $\Delta OCS_2$  are congruent. The analysis of segment relationships revealed the following:

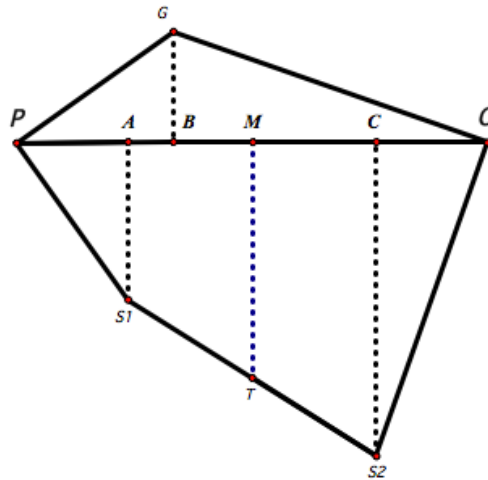
$$AM = PM - PA = \frac{1}{2}PO - PA = \frac{1}{2}(PO - 2PA) = \frac{1}{2}(PO - PA - OC) = \frac{1}{2}AC$$

Hence, M is also the midpoint of segment AC.

Quadrilateral  $ACS_1S_2$  is a trapezoid since  $AS_1$  and  $CS_2$  are parallel. Using the mid-segment theorem, the students obtained:

$$MT = \frac{1}{2}(AS_1 + CS_2) = \frac{1}{2}(PB + BO) = \frac{1}{2}PO = PM$$

Hence, this showed that the location of the gallows is not important to locate the treasure, but dependent by the location of the pine and oak tree. Moreover, the treasure is located exactly in the middle of the two trees. In order to find the location, Marco would have to walk from P or O towards M, and then turn right or left and walk the same number of steps to reach the treasure (see Figure 3).



**Figure 3** Geometrical solution of the treasure island problem

**Solution using vector algebra**

This approach follows as outlined in the first approach, however this time the students operated with vectors. The following vectors were defined:

$$\vec{T} = \frac{1}{2}(t + (-t), x_1 - s + x_1 + s) = (0, x_1).$$

$$\vec{P} = (-x_1, 0), \vec{O} = (x_1, 0)$$

Using dot product, the coordinates of the two spikes can be easily be calculated:  $S_1=(t, x_1-s)$  and  $S_2=(-t, x_1+s)$ . The vector  $\vec{T}$  can be then easily found by adding the vectors:

$$\vec{T} = \frac{1}{2}(t + (-t), x_1 - s + x_1 + s) = (0, x_1).$$

### ***Solution using complex numbers***

Similar to before, we lay down a plane, but a complex one. Hence, the island is lying in a plane of complex numbers. The real axis passes through points P and O, whereas the imaginary axes passes through a point half way between the points P and O. Hence, the point P (-1, 0) and point O (1, 0), or as a complex number  $-1+0i$  and  $1+0i$ , respectively. Let  $\Gamma$  be the location of the gallows, some arbitrary complex number:  $\Gamma=a+bi$ . Using the distance formula between two complex numbers, the distance between the oak tree and the gallows, and the pine tree and the gallows can be calculated. Distance between the pine tree and the gallows can be denoted by  $1-\Gamma$ , whereas between the oak tree and the gallows as  $-1-\Gamma=-(1+\Gamma)$ . To find the location of the spikes, we need to multiply the distance by  $-i$  and by  $i$ , to allows for a clockwise and counterclockwise rotation in the complex plane, respectively. Therefore, the first spike  $S_1=i(1-\Gamma)-1=i-i\Gamma-1$  and the second spike  $S_2=(-i)(-1-\Gamma)+1=i+i\Gamma+1$ . Taken that the treasure is located halfway between the two spikes, we use the midpoint formulas for two complex numbers. We get:

$$\frac{1}{2}(i-i\Gamma-1+i+i\Gamma+1)=\frac{1}{2}(2i)=i$$

Hence, the location of the gallows is not relevant to locating the treasure, which is located at a fix value  $i$  in the complex plane – that is, the point (0, 1) and at the midpoint of the oak and pine tree.

### **4.5 Data analysis**

For the purpose of this study, multiple stages of analysis, as suggested by Patton (2002) were conducted using both inductive and deductive approach, which is explained in the following paragraphs.

To answer the first research sub-question the interviews, participants' booklet with the problem of interest, and researcher's field notes were transcribed. This was followed by using inductive methods to analyze the data. I analyzed the data for convergence or determining which pieces of data were similar using inductive analysis that allows construction of themes or theories that are "grounded" in the data. When using inductive analysis, I focused on creating codes and categories from the data, developing or enhancing theory during the act of analysis and the use of constant comparative method during analysis of the data. Using the constant comparative method, I categorized all data that consisted of comparing and generating categories, integrating categories, and delimiting the theory to help illuminate common themes across cases and within cases. In addition, I used observations of the participants in the course to confirm or disconfirm my hypotheses about their outtake on the nature of their problem solving in a technology environment. To answer the second research sub-question, I transcribed the interview with the course instructor. Afterwards I followed the same procedure as described for the first research sub-question.

With respect to the third research question, I analysed participants' booklet using deductive methods based on the outlined three-model framework by Salomon and Perkins (2005). I first noted what different DGS affordances were used within each episode. These included drawing objects, figures and auxiliary lines, constructing objects, using editing, transformational and measurement tools, calculations, dragging, tracing and locus. In additional step, I looked at how the availability of DGS influenced the decisions participants made with respect to solving the problem, the role that technology played in the mathematical work, and the level of thinking needed for students to appropriately use the technology when problem solving. On the basis of these considerations, I made an assessment of how the availability and use of technology affected the cognitive processing during problem solving.

To assure validity and reliability of the study, I used triangulation, thick rich description, and the audit trail. When talking of triangulation, I used triangulation of sources and analyst triangulation. Triangulation of qualitative data sources involved comparing observations with interviews and checking for the consistency of what participants said during the think-aloud session and during the interview session. Finally with regard to analyst triangulation, I used member checking and another expert. The inter-rater reliability was high and equaled 97% using the formula recommended by Miles and Hubermann (1994). Employing the procedures mentioned above ensured trustworthiness and rigor.

## 5 Results

In the following I describe the three perspectives as they pertain to the research sub-questions. The first perspective is that of the participants, whose portrait includes reflection on their learning (e.g., mathematical activity, use of technology). The second perspective is the perspective of the instructor itself, whose portrait includes his own views about his students' mathematical activity in DGS. Lastly, the third perspective is that of a researcher (author of this paper) interested in understanding how DGS, influence student problem solving processes (e.g., relationship between the technology and the user).

### 5.1 Importance of the DGS when faced with mathematical task in a technological context: Preservice and inservice teachers' perspective

The first perspective sought to explain participants' perceptions of the importance of DGS when solving the Gamow's treasure problem. Both groups expressed that DGS was an important and useful tool during problem solving centered on these qualities: problem solving activities and processes, visualization, speed, and accuracy.

They felt strongly about DGS helping them during various stages of problem solving, most notably during exploration. It gave them the opportunity to explore the problem with relative ease; to manipulate the figure and monitor the change that helped gather relevant information that they used when working through problem solving space. DGS also helped assess their actions and conjectures and deciding whether to refine, revise or abandon a

particular perspective. Some participants talked about the component of better understanding the problem. For instance, one preservice teacher, held a similar perception adding that exploring the problem aided her to better understand the problem, and access knowledge and strategies. She then considered if these were relevant to the problem or not. In addition such behaviors aided organizing that knowledge in moving successfully towards a solution. Verifying that the answer was an appropriate solution and examining the path to obtain it using DGS's Measurement Tool was important for participants and was used also to double-check their actions. The participants perceived the importance of being able to represent not only the problem, but also an idea with DGS. Making a representation of the problem was held important in developing an understanding of the problem, examining relationships between conditions and the goals of the problem as well as considering and selecting a choice of perspective.

During the final interview one participant stated, "The ease of discovery provides an excellent resource." He viewed DGS helpful in allowing him to outline the solution and strategy and test them cycling back making the process of problem solving fluid and less discouraging. There were several admissions that a solution to a problem might have not been possible or successful without the use of DGS. Some participants seemed to perceive that for the treasure island problem, problem solving without the use of DGS would have been time consuming, detracting them from the purpose of the problem and the problem solving process. Instead DGS allowed them to stay organized and focused.

Though participants perceived many attributes of DGS helpful during their problem solving, they viewed it as hindrance as well. For instance, some preservice teachers stated that technology can be addictive to the extent of not planning appropriately, assessing and monitoring their actions. Some said that using dynamic features of the software was detrimental to quality of their reasoning and outlined plans.

## **5.2 Patterns of behavior – Lecturer's perspective**

As a result of open problem solving approach in DGS, different aspects with respect to problem solving, technology, and mathematics were observed by the course instructor when solving the Gamow's hidden treasure problem. Below I outline different themes and provide a description for each of them.

- *Model and solve the problem.* The first step involved translating the mathematical statement into a diagram. This allowed students to assess whether they understand the problem. After representation was produced, they modeled the problem to find a plausible solution. By using affordances of the DGS, they were able to conjecture, test, refine or refute these conjectures about a solution to the problem. Whereas finding a plausible conjecture was done with ease, proving that plausible conjecture was not. However, based on the different knowledge level of students, different approaches were found. Whereas the solution using Euclidean geometry and

coordinate geometry can be used in with high-school students, the solution using vector algebra and complex number is more appropriate for university students.

- *Apply knowledge to model the problem situation.* Modeling the problem situation in a DGS required a big effort. At first many students modeled the problem on paper-and-pen before deciding to move to DGS. While the former required very little effort, or did not cause any difficulties, the latter proved to be a challenge. Here knowing and understanding the relationships between the geometric objects and knowing how to construct them in a DGS was paramount. Thus, creating interactive representation of the problem allowed students to apply, consolidate and expand their knowledge of DGS.
- *Generate, test, refine, and justify mathematical conjectures.* As already noted, after the mathematical model was established, the students engaged in solving the problem. By using the dragging modalities, the participants could formulate their conjectures, and based on the visual feedback refine or refute them. Hence, this process was dynamic and cyclic involving generation, testing, refinements and in the end justification of the conjecture, as suggested by Wilson et al. (1993).
- *Use DGS to establish mathematical relationships.* In order to model a mathematical situation and its relationships, the participants needed to translate the verbal statements into an interactive diagram.
- *Develop personal habits of mind, such as perseverance, persistence, disposition, motivation to solve the problem.* When solving open problems by paper-and-pen students gave up fast because they could not explore the problem. Having a static diagram did not allow each participant with the opportunity to tackle the problem, as mathematical ideas needed to solve the problem are not immediately apparent. When problem solving in a DGS, they were more enthusiastic and persistent in solving the problem. By exploring the problem solving they were able to use the features of the environment and go further in their problem solving space. Hence, DGS supported the strategies that were not necessarily available on paper-and-pen. For that reason, they could develop a better understanding of the problem, explore it, focus on the essence on the problem, which then positively influenced their affective domain.
- *Experience joy, proud when being engaged in genuine discovery.* One of observed behaviors was certainly joy of being engaged in a discovery problem solving. These students had little, if any, opportunities to engage in problem solving activities, but rather were used to textbook problems. Open problems allowed students to discover plausible solutions, test it and experience the joy of discovering the solution to the problem.

### **5.3 Association of different activities and the technology use – Researcher’s perspective on the different technology effects**

Here I outline different uses of technology with respect to different cognitive behaviors and heuristics. Salomon and Perkins’s (2005) framework for technological effects on intelligence was used to examine the relationship between technology use in this study and problem solving processes and is presented in the following section.

#### ***Effects with technology***

Effects with technology, where the participant delegated certain jobs to be carried out by the DGS were seen with all participants. The CONSTRUCT functions allowed the participants to make a representation of the problem and quickly add secondary elements, lines, segments, rays, midpoints in the figure, and construct perpendicular lines. The MEASURE functions allowed participants to measure the distances, find the location of the treasure and discover that its location is invariant. Finding the location on paper and pen would have taken a longer time and would have distracted the participants from focusing on problem solving process, rather than on the procedures itself. Salomon and Perkins (2005) attend to this as freeing “the user from distractions of lower-level cognitive functions” (p. 74). Such affordance in conjunction with solving a problem solving task exemplifies effects with technology.

#### ***Effects of technology***

Effects of technology, where a participant’s understanding or ability with respect to mathematics (independent of the DGS) had changed through extended tool use after a period of using it, was more difficult to observe. I was able to discern some of these on the basis of the final interview, that took place at the end of the study, which I could characterize as either positive or negative. On the one hand technology was used creatively to enhance the user’s empowerment, and as an extension of the user, namely, where the user drew on his or hers technological expertise as an integral part of their mathematical thinking. On the other hand the way that the technology got used impoverished the system – they were subservient to the technology and the relationship was one of dependence. Tall (1989) added that behaviors characterized by loss of autonomy in the problem solving process during which common sense for using the system is ignored, so called the “authority of the machine”, is most often observable by those who have limited knowledge of the technology in use. The DGS seemed to at times to encourage the lack of monitoring behaviors that some researchers (Olive & Makar 2010; Schoenfeld 1992) describe as characteristic, when students work on unfamiliar problems. Thus, knowing how each system operates and what affordances it possesses is a prerequisite for a goal-oriented, meaningful and effective use of any technology during problem solving. However, the final interview revealed that for those students their problem solving model became apparent, and reflected critically on it:

Interviewer: *What did you learn through this technology problem solving experience?*

Interviewee: *It taught me a lot about how I think about mathematics. I realized some of my problem-solving techniques that I typically employ and it allows me to analyze the pros and cons of my approaches. I find that while I supported technology use in almost all problem-solving contexts before this study, I can now determine which problems can use technology most effectively. I am now of the opinion that certain problems are more geared towards technology, while in others, it can be a hindrance.*

Thus, recognition of faulty usage, namely an excessive dependence on the tool, causing limited engagement in problem solving processes, was an effect that remained long after a period of using technology.

### ***Effects through technology***

Effects through technology could be identified through the nature of the problem solving process the participants engaged in, which was fundamentally different from those that would have been used prior to the introduction of the DGS. The task presented here did not have to be solved using the DGS, but on paper and pen. Using DGS, however, allowed the majority of the participants to represent and think about the mathematical context in ways that previously were not possible. One of the key attributes of the DGS that allow new mathematical practices is to visually represent geometrical invariants by using dragging. Dragging modalities has shown to be conducive to knowledge construction while problem solving. This is shown in the following excerpt.

Wes: *(Moves the gallows point). The treasure doesn't move no matter where the gallows is. Now I need to justify why does the treasure doesn't move. Why is it always in the same location? How do I prove that something is an invariance?*

Interviewer: *How did you resolve this?*

Wes: *Well, I tried giving the Euclidean geometry and I did several sketches trying to, just trying to figure it out... Basically I worked on the Euclidean one day, uhm, came back to it the next day, tried another sketch, tried working with it, and I was like I am not getting anywhere... So then I went to vectors but first I established a coordinate system.*

Interviewer: *Why?*

Interviewee: *Because I need somehow, I need to figure out a way to establish relationships between one another, so I thought I need to have a coordinate system that is fixed, so that I can determine how the trees are related to one another, and how the gallows is related to the two trees, and then eventually locate the treasure.*

Here the participant engaged in conjecturing-testing-analyzing sequence of steps. By wandering dragging<sup>2</sup> (Arzarello, Micheletti, Olivero et al., 1998) he created contrasting experiences, which further allowed separating critical features relevant to the problem and

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<sup>2</sup> Wandering dragging, that is moving the basic points on the screen more or less randomly, without a plan, in order to discover interesting configurations or regularities in the figures.

with that developing a better understanding of the current problem solving situation. Visualization of the problem triggered organizing their knowledge when seeking relationships between the conditions and the goals of the problem. Simultaneously engaging in reasoning and monitoring activities, a conjecture was formulated and tested/implemented. This result support results from researchers (e.g., Leung, Chan, & Lopez-Real, 2006; Mariotti, 2006) about the effectiveness technology in developing and supporting visual problem solving methods, and through the feedback the system gives theoretical thinking gets enhanced. The feedback offers opportunities to engage in different reasoning processes, such as proving or abstracting as a result of a reflection on an outcome of an action, as I presented in the summary above. Thus, DGS was an effective tool to aid the participants in monitoring and controlling their problem solving progress; when inadequate solution is obtained, it signalizes the use to rethink his thinking process, and refine or refute a conjecture, an approach, a strategy, etc. Moreover, visualization and feedback allows the user to access mathematical knowledge by integrating a symbolic with motor-perceptive approach, and as a result learn on the basis of doing, moving, and seeing cycle in the problem solving process (Chiappini & Bottino, 1999).

To sum up, the problem solving behaviours included, but were not limited to, conjecturing before generalizing, exploring before proving, visualizing multiple cases and through it developing a better understanding for the problem before deciding on a final formulation of a statement. Hence, instead were the processes reversed from “typical” result’s formalization.

## 6 Conclusions

Past and current reforms in mathematics advocate that problem solving becomes a part of mathematics classroom. Unfortunately, achieving this goal presented to be a challenging one. Geometry courses are frequently presented as completed axiomatic-deductive system. Rarely they have chances to engage in open problems instigating geometry being taught as a finished product. In this paper I have tried to illustrate that working in a DGS allow students to go beyond the memorization and execution of algorithms to solve open problems. DGS got used to model and solve complex problems, that is, problems requiring flexibility, creativity in thinking and non-standard and non-algorithmic procedures. Students engaged in open, practical and investigative tasks and through it helped students synthesize their mathematical knowledge, apply and adapt a plethora of paper-pen and DGS problem solving strategies, and monitor their problem solving processes. Teaching pre- and inservice teachers to become better problem solvers is a challenging task. However, technological environments such as DGS may facilitate this mission. The presence of DGS when problem solving may also enable not only teachers but also students to understand, approach, and solve a wider variety of problems that would not be possible without it. Providing three different perspectives helps understand the complexity of the phenomenon. I cannot hope to convey what I have seen, which is less that what was there,



but I hope I have shown three ways of seeing what happens in a technological problem solving instruction, perhaps offering possibilities for mathematics educators engaged with students' problem solving.

The new and emerging technologies continually transform the mathematics classroom and redefine the ways mathematics can be taught and how mathematics is learned. They make a dynamic impact on the way we teach, learn, and problem solve. Here new goals or new content will not be typified, but rather allow new ways for old goals to be reached and content to be understood, as well as new type of problems to be (more easily) solved. Mathematics educators (teachers, university lecturers) need to find or develop such open problems whose problem solving paths can be facilitated by the appropriate use of DGS. This would allow students to become also better problem solvers and experience genuine problem solving and the joy when the discovery occurs.

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